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A Canonical Description of the Plancherel Measure for a General Two-Step Free Nilpotent Lie Group.

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A CANONICAL DESCRIPTION OF THE PLANCHEREL MEASURE FOR A GENERAL TWO-STEP FREE NILPOTENT LIE GROUP

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Abstract

In this work on $\mathfrak{g} = \mathcal{F}_{n,2}$, the free 2-step nilpotent Lie algebra on n generators, we use the group of automorphisms to give a basis-free description of the Fourier Inversion Formula, thereby generalizing and strengthening an example discussed by Corwin and Greenleaf.

In the first charter, Introduction, we begin with a brief survey of traditional viewpoint related to this dissertation, then discuss Example 4.3.14 in Corwin & Greenleaf's book. It demonstrates how two different bases for $\mathcal{F}_{3,2}$ lead to different inversion formulas. But the third "more" invariant formula describes Plancherel measure on a support expressed in terms of rotations, dilations, and translations. Actually it is not canonical since it still depends on choices of bases for $\mathcal{F}_{3,2}$. Our goal is to re-describe Plancherel measure on a support expressed in terms of $\mathrm{Aut}^*(\mathfrak{g})$. We accomplish this in the following two chapters.

The second chapter provides a procedure for re-parametrizing the family of generic orbits by establishing a one-to-one correspondence between the maximum-dimensional orbits and the quotient space $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$.

The third chapter provides background material about relatively invariant measures. Then we prove that Plancherel measure, modeled on the double coset space $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$, is the essentially unique relatively invariant measure corresponding to a specific homomorphism.

The fourth chapter demonstrates that there does not exist an $\overline{\mathrm{Aut}^*(\mathfrak{g})}$-invariant measure on the double coset space $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$ for the example $\mathfrak{g} = \mathcal{F}_{3,2}$.

Our explicit calculations for $\mathcal{F}_{3,2}$ are done in Chapter 5 and are in agreement with the results of the first four chapters. We start by finding an almost global

coordinate patch for $\mathrm{Ad}_G\backslash\mathrm{Ad}_G\cdot\mathrm{Stab}^*(Z_3^*)$, then use this patch to construct a right and left Haar measure on this quotient space. Thus we get its modular function δ. A similar process applies to $\mathrm{Ad}_G\backslash\mathrm{Aut}(\mathfrak{g})$, and we obtain its modular function Δ. Hence the ratio of Δ to δ is the restriction of any modular function for any relatively invariant measure on the double coset space $\mathrm{Ad}_G\backslash\mathrm{Aut}(\mathfrak{g})/\mathrm{Stab}^*(Z_3^*)$. Furthermore, we find a cross-section $\mathbf{X}$ for $\mathrm{Ad}_G\backslash\mathrm{Aut}(\mathfrak{g})/\mathrm{Stab}^*(l_0)$ with general $l_0 \in \mathfrak{g}^*_{Max}$. Then we use $\mathbf{X}$ to verify the relative invariance for the measure corresponding to that given in the example of Corwin and Greenleaf for $\mathfrak{g} = \mathcal{F}_{3,2}$.

In Chapter 6, we illustrate three general properties for $\mathcal{F}_{n,2}$ and one additional result for linear algebra. These properties are used in Chapter 5.

1. Introduction

We begin with some general remarks about the subject matter of this dissertation. Harmonic analysis is a very rich subject which can be viewed from many perspectives. One traditional viewpoint is that harmonic analysis permits the recovery of a function from its Fourier transform: For each Schwartz function $f \in \mathcal{S}(\mathbf{R})$,

$$f(e) = \int_{\mathbf{R}} \widehat{f}(\alpha) d\alpha$$

One can readily recover also $f(x)$ for general x simply by replacing f in the integral by its translation by x. If one does this, then

$$f(x) = \int_{\mathbf{R}} \widehat{f}(\alpha) e^{2\pi i x \alpha} d\alpha$$

The Real numbers $\mathbf{R}$ are an abelian group under addition. When one investigates this process of Fourier Inversion for general locally compact topological groups, or for Lie groups, it becomes necessary to replace the 1-dimensional characters $\chi_\alpha(x) = e^{2\pi i \alpha x}$ by the equivalence classes of irreducible unitary representations in Hilbert space. The collection of these classes is denoted $\widehat{G}$. If the group G is neither compact nor abelian, then $\widehat{G}$ typically includes the classes of many infinite dimensional irreducible representations. There is an abstract Plancherel theorem (due to Segal) which establishes that

$$f(e) = \int_{\widehat{G}} \mathrm{Tr}(\pi_f) d\mu(\pi)$$

for all suitable functions f, such as Schwartz functions on a Lie group. Here $\pi_f = \int_G f(g)\pi_g dg$, and the trace of this operator plays the role of $\widehat{f}(\alpha)$ in classical Fourier inversion.

In order to carry out this Fourier inversion in a meaningful way on particular classes of Lie groups, it is necessary first to know the representations in $\widehat{G}$ and to know explicitly the Plancherel measure and its support within $\widehat{G}$. The representations of Lie groups are often constructed using the classical process of *Induced Representations*, developed by Schur for finite groups and greatly extended by George Mackey to all locally compact topological groups in the 1940's and 1950's. (See [7]) Much effort was expended upon the identification of $\widehat{G}$ for general nilpotent Lie groups until this was finally achieved by A. A. Kirillov in his celebrated dissertation [6] under the direction of I. M. Gelfand. Kirillov showed that if G is nilpotent, then every element of $\widehat{G}$ is induced by a one-dimensional character of a suitable subgroup $M \subset G$. If $l \in \mathfrak{g}^*$ one defines the character $\chi_l(m) = e^{2\pi i l(\log m)}$ and χ will induce an irreducible unitary representation of G if and only if M has maximal dimension so as to make χ_l a genuine character. Moreover, the unitary equivalence classes in $\widehat{G}$ are in one-to-one correspondence with the coadjoint orbits of G, also called Kirillov orbits. Thus $\widehat{G}$ can be identified with $\mathfrak{g}^*/\mathrm{Ad}^*(G)$. The support of the Plancherel measure μ was shown to be a very large collection of orbits, whose union constitutes a Zariski open subset of $\mathfrak{g}^*$. All orbits in the support of μ are of the greatest possible dimension, but not every such orbit need be in that support. One of the features of classical harmonic analysis which is lost in the study of Lie groups is that $\widehat{G}$ lacks a group structure. (For abelian groups, $\widehat{G}$ is a group.) In this dissertation we address that lack. We show that at least for certain types of nilpotent Lie groups one can create a model of the Plancherel measure on a quotient space of the automorphism group, $\mathrm{Aut}(G)$. When we do this, we show that the Plancherel measure μ is the *essentially unique relatively invariant measure* corresponding to the homomorphism which maps each automorphism

$A \to |\det(A)|$. The idea arose out of our efforts to understand better an example of Corwin and Greenleaf, presented in their book [1].

It is an awkward feature of nilpotent harmonic analysis that the description of the Plancherel measure, and even of the generic orbits on which it is supported, is dependent on the choice of an arbitrary strong Malcev basis. Corwin & Greenleaf illustrate this with the following Example 4.3.14 in [1]:

Let $\mathfrak{g} = \mathcal{F}_{3,2}$, $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$, and let $\mathfrak{g}^*$ be the linear dual space of $\mathfrak{g}$. Pick a basis $\{\overline{Y}_1, \overline{Y}_2, \overline{Y}_3\}$ for $\mathfrak{g}/\mathfrak{g}_2$, then choose $Y_i \in \overline{Y}_i$, $\forall i = 1, 2, 3$. Let

$$Z_1 = [Y_2, Y_3],\ \ Z_2 = [Y_3, Y_1],\ Z_3 = [Y_1, Y_2],$$

which are independent of the choice of $Y_i \in \overline{Y}_i$. Then $\mathcal{B} := \{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$ is a strong Malcev basis of $\mathfrak{g}$. Let $\mathcal{B}^* := \{Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*\}$ be the dual basis of $\mathcal{B}$ in $\mathfrak{g}^*$, so that $\mathcal{B}^*$ is a Jordan-Hölder basis of $\mathfrak{g}^*$. Let $\mathfrak{g}_1 = \mathrm{Span}_{\mathbf{R}}\{Y_1, Y_2, Y_3\}$. We know $\mathfrak{g}_2 = \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2, Z_3\}$. Let $\mathfrak{g}_1^* = \mathrm{Span}_{\mathbf{R}}\{Y_1^*, Y_2^*, Y_3^*\}$ and $\mathfrak{g}_2^* = \mathrm{Span}_{\mathbf{R}}\{Z_1^*, Z_2^*, Z_3^*\}$.

Corwin & Greenleaf describe the Plancherel measure with respect to this strong Malcev basis $\mathcal{B}$ of $\mathfrak{g}$ as follows. $l = \sum_{i=1}^3 \alpha_i Z_i^* + \sum_{i=1}^3 \beta_i Y_i^*$ is generic with respect to this basis if and only if $\alpha_3 \neq 0$, and the Pfaffian for this basis $\mathcal{B}$ is given by $\mathrm{Pf}^2(l) = \det \overline{B}_l = \alpha_3^2$. The Fourier Inversion Theorem says that for $f \in \mathcal{S}(G)$, the Schwartz functions on G,

$$f(e) = \int_{\mathbf{R}^4} |\alpha_3| \mathrm{Tr}\pi_{\alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^* + \beta_3 Y_3^*}(f) d\alpha_1 d\alpha_2 d\alpha_3 d\beta_3 \tag{1.1}$$

On the other hand, if we take the strong Malcev basis to be $\{Z_3, Z_2, Z_1, Y_3, Y_2, Y_1\}$, then l is generic if and only if $\alpha_1 \neq 0$. And the corresponding Pfaffian is given by $\mathrm{Pf}^2(l) = \alpha_1^2$. The Fourier Inversion Theorem now says that

$$f(e) = \int_{\mathbf{R}^4} |\alpha_1| \mathrm{Tr}\pi_{\alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^* + \beta_1 Y_1^*}(f) d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 \tag{1.2}$$

Why do these two different formulas, (1.1) and (1.2), give the same result f(e)? To understand this situation, Corwin & Greenleaf describe how the group $SO(3, \mathbf{R})$ acts on $\mathfrak{g}$ (and G) as a group of automorphisms. We discuss on the following pages how their explanation works. Then we will give a preview of how we improve and generalize what they have done.

We decompose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a sum of vector spaces. Notice that $\mathfrak{g}_1$ is not a Lie subalgebra of $\mathfrak{g}$. Since $\mathcal{B}_1 := \{Y_1, Y_2, Y_3\}$ is a basis of $\mathfrak{g}_1$, we regard $\mathfrak{g}_1$ as isomorphic to $\mathbf{R}^3$ by the map $Y \longmapsto [Y]_{\mathcal{B}_1}$. Similarly, $\mathfrak{g}_2$ is regarded as isomorphic to $\mathbf{R}^3$ relative to the basis $\mathcal{B}_2 = \{Z_1, Z_2, Z_3\}$.

For each $\sigma \in SO(3, \mathbf{R})$, we define the map

$$(\sigma \otimes \sigma) : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \longrightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2 \text{ by } (Y, Z) \longmapsto (\sigma Y, \sigma Z)$$

Then we claim that $\sigma \otimes \sigma$ is an automorphism of $\mathfrak{g}$.

It suffices to show that $[\sigma Y_i, \sigma Y_j] = \sigma([Y_i, Y_j])$, for all pairs (i, j), $1 \leq i, j \leq 3$. We proceed to show this as follows.

For each $\sigma \in SO(3)$, we still denote the matrix of σ relative to the standard basis of $\mathbf{R}^3$ by σ. Let $\sigma = \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix}$. By the definition of SO(3), we have $(\sigma^{-1})^t = \sigma$. Then $(\frac{1}{\det \sigma}(\mathrm{adj}\sigma))^t = \sigma$, and hence $(\mathrm{adj}\sigma)^t = \sigma$ since $\det \sigma = 1$. Thus we get the identity

$$\begin{bmatrix} \det\begin{pmatrix} p_5 & p_8 \\ p_6 & p_9 \end{pmatrix} & -\det\begin{pmatrix} p_2 & p_8 \\ p_3 & p_9 \end{pmatrix} & \det\begin{pmatrix} p_2 & p_5 \\ p_3 & p_6 \end{pmatrix} \\ -\det\begin{pmatrix} p_4 & p_7 \\ p_6 & p_9 \end{pmatrix} & \det\begin{pmatrix} p_1 & p_7 \\ p_3 & p_9 \end{pmatrix} & -\det\begin{pmatrix} p_1 & p_4 \\ p_3 & p_6 \end{pmatrix} \\ \det\begin{pmatrix} p_4 & p_7 \\ p_5 & p_8 \end{pmatrix} & -\det\begin{pmatrix} p_1 & p_7 \\ p_2 & p_8 \end{pmatrix} & \det\begin{pmatrix} p_1 & p_4 \\ p_2 & p_5 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix}$$

It follows that

$$\begin{aligned} [\sigma Y_1, \sigma Y_2] &= [p_1Y_1 + p_2Y_2 + p_3Y_3, p_4Y_1 + p_5Y_2 + p_6Y_3] \\ &= \begin{vmatrix} p_2 & p_5 \\ p_3 & p_6 \end{vmatrix} Z_1 - \begin{vmatrix} p_1 & p_4 \\ p_3 & p_6 \end{vmatrix} Z_2 + \begin{vmatrix} p_1 & p_4 \\ p_2 & p_5 \end{vmatrix} Z_3 \\ &= p_7Z_1 + p_8Z_2 + p_9Z_3 \\ &= \sigma Z_3 \\ &= \sigma([Y_1, Y_2]) \end{aligned}$$

$$\begin{aligned} [\sigma Y_2, \sigma Y_3] &= [p_4Y_1 + p_5Y_2 + p_6Y_3, p_7Y_1 + p_8Y_2 + p_9Y_3] \\ &= \begin{vmatrix} p_5 & p_8 \\ p_6 & p_9 \end{vmatrix} Z_1 - \begin{vmatrix} p_4 & p_7 \\ p_6 & p_9 \end{vmatrix} Z_2 + \begin{vmatrix} p_4 & p_7 \\ p_5 & p_8 \end{vmatrix} Z_3 \\ &= p_1Z_1 + p_2Z_2 + p_3Z_3 \\ &= \sigma Z_1 \\ &= \sigma([Y_2, Y_3]) \end{aligned}$$

$$
\begin{aligned}
[\sigma Y_3, \sigma Y_1] &= [p_7 Y_1 + p_8 Y_2 + p_9 Y_3, p_1 Y_1 + p_2 Y_2 + p_3 Y_3] \\
&= -\begin{vmatrix} p_2 & p_8 \\ p_3 & p_9 \end{vmatrix} Z_1 + \begin{vmatrix} p_1 & p_7 \\ p_3 & p_9 \end{vmatrix} Z_2 - \begin{vmatrix} p_1 & p_7 \\ p_2 & p_8 \end{vmatrix} Z_3 \\
&= p_4 Z_1 + p_5 Z_2 + p_6 Z_3 \\
&= \sigma Z_2 \\
&= \sigma([Y_3, Y_1])
\end{aligned}
$$

Thus we have proved that $\sigma \otimes \sigma$ is in $\mathrm{Aut}(\mathfrak{g})$.

Similarly, we regard $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ as each isomorphic to $\mathbf{R}^3$ relative to coordinates in the bases $\{Y_1^*, Y_2^*, Y_3^*\}$ and $\{Z_1^*, Z_2^*, Z_3^*\}$ respectively. For each $\sigma \in \mathrm{SO}(3)$, we define the dual σ^* of σ on $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ by its inverse transpose. Since $\sigma \otimes \sigma \in \mathrm{Aut}(\mathfrak{g})$, so is its inverse $(\sigma \otimes \sigma)^{-1} = \sigma^{-1} \otimes \sigma^{-1}$. Define the dual of $\sigma \otimes \sigma$ by its inverse transpose. So we have $(\sigma \otimes \sigma)^* = \sigma^* \otimes \sigma^*$.

Now we give $\mathfrak{g}_1^*$ an inner product and a corresponding Euclidean metric by making Y_1^*, Y_2^*, and Y_3^* orthonormal, and similarly for $\mathfrak{g}_2^*$ by making Z_1^*, Z_2^*, and Z_3^* orthonormal. Define a linear map $A : \mathfrak{g}_2^* \longrightarrow \mathfrak{g}_1^*$ by $A(Z_i^*) = Y_i^*$. For every unit vector $\omega \in \mathfrak{g}_2^*$, let $\omega = \alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^*$. Then there exists $\sigma \in \mathrm{SO}(3)$ such that $\omega = \sigma^* Z_3^*$. It follows that

$A\omega = A(\alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^*) = \alpha_1 Y_1^* + \alpha_2 Y_2^* + \alpha_3 Y_3^* = \sigma^* Y_3^*$. So

$r\omega + tA\omega = r\sigma^* Z_3^* + t\sigma^* Y_3^* = (\sigma \otimes \sigma)^*(rZ_3^* + tY_3^*)$, $\forall r > 0$, $\forall t \in \mathbf{R}$.

For $l_0 = rZ_3^* + tY_3^*$, its radical $\mathfrak{r}_{l_0} = \mathfrak{g}_2 \oplus \mathbf{R}Y_3$ is an ideal in $\mathfrak{g}$. By Theorem 3.2.3. in [1] we know that $\mathcal{O}_{l_0} = l_0 + \mathfrak{r}_{l_0}^{\perp} = l_0 + \mathrm{Span}_{\mathbf{R}}\{Y_1^*, Y_2^*\}$. For $l = r\omega + tA\omega = (\sigma \otimes \sigma)^*(rZ_3^* + tY_3^*)$, its radical $\mathfrak{r}_l = \mathfrak{r}_{(\sigma\otimes\sigma)^* \cdot l_0} = (\sigma \otimes \sigma)\mathfrak{r}_{l_0} = \mathfrak{g}_2 \oplus \mathbf{R}(\sigma Y_3)$ is still an ideal of $\mathfrak{g}$. Then $\mathcal{O}_l = l + \mathfrak{r}_l^{\perp} = l + \mathrm{Span}_{\mathbf{R}}\{\sigma^* Y_1^*, \sigma^* Y_2^*\} = r\omega + tA\omega + (A\omega)^{\perp}$ where $(A\omega)^{\perp}$ is the plane $\mathrm{Span}_{\mathbf{R}}\{\sigma^* Y_1^*, \sigma^* Y_2^*\}$ orthogonal to $A\omega$ in $\mathfrak{g}_1^*$.

Let $\mathcal{O}_{Max} = \{\mathcal{O}_l : \dim \mathcal{O}_l \text{ is } maximal\}$, and $\mathfrak{g}^*_{Max} = \{l : \dim \mathcal{O}_l \text{ is } maximal\}$. We claim that

$$\{r\omega + tA\omega | r > 0,\ t \in \mathbf{R},\ \omega \in \mathfrak{g}_2^*,\ |\omega| = 1\}$$

is a cross-section of $\mathcal{O}_{Max}$.

Since $l \in \mathfrak{g}^*_{Max}$ if and only if $l|_{\mathfrak{g}_2} \neq 0$, there always exist unique $r > 0$ and unit vector $\omega \in \mathfrak{g}_2^*$ such that $l|\mathfrak{g}_2 = r\omega$. For each $t \in \mathbf{R}$, $tA\omega + (A\omega)^\perp$ is a two-dimensional plane parallel to $(A\omega)^\perp$ in $\mathfrak{g}_1^*$. So

$\mathfrak{g}_1^* = \dot{\bigcup} \{tA\omega + (A\omega)^\perp : \forall t \in \mathbf{R}\}$, and hence there is a unique $t \in \mathbf{R}$ such that $l|\mathfrak{g}_1 \in tA\omega + (A\omega)^\perp$. It follows that $l \in r\omega + tA\omega + (A\omega)^\perp = \mathcal{O}_{r\omega+tA\omega}$. Therefore $\{r\omega + tA\omega | r > 0,\ t \in \mathbf{R},\ \omega \in \mathfrak{g}_2^*,\ |\omega| = 1\}$ is a cross-section for $\mathcal{O}_{Max}$, as claimed.

For $\omega_0 = Z_3^*$, $l_0 = r\omega_0 + tA\omega_0$ lies on the orbit $\mathcal{O}_{l_0} = r\omega_0 + tA\omega_0 + \mathbf{R}Y_1^* + \mathbf{R}Y_2^*$. And relative to the basis $\{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$, the Pfaffian of l_0 is given by $Pf^2(l_0) = r^2$. Then for every $f \in \mathcal{S}(\mathrm{G})$,

$$\mathrm{Tr}\pi_{r\omega_0+tA\omega_0}(f) = \int_{\mathbf{R}^2} \frac{1}{r} (f \circ \exp)\hat{}\,(r\omega_0 + tA\omega_0 + \beta_1 Y_1^* + \beta_2 Y_2^*) d\beta_1 d\beta_2.$$

Hence

$$\int_{\mathbf{R}} \mathrm{Tr}\pi_{r\omega_0+tA\omega_0}(f) dt = \int_{\mathfrak{g}_1^*} \frac{1}{r} (f \circ \exp)\hat{}\,(r\omega_0 + \lambda) d\lambda$$

where $d\lambda$ is normalized Lebesgue measure defined relative to the basis $\{Y_1^*, Y_2^*, Y_3^*\}$ on $\mathfrak{g}_1^*$.

Now for each $\sigma \in \mathrm{SO}(3)$, since $\sigma \otimes \sigma \in \mathrm{Aut}(\mathfrak{g})$, it follows that

$$\pi_{(\sigma\otimes\sigma)^*(r\omega_0+tA\omega_0)} \cong \pi_{r\omega_0+tA\omega_0} \circ (\sigma \otimes \sigma)^{-1}$$

, and hence

$$\pi_{(\sigma\otimes\sigma)^*(r\omega_0+tA\omega_0)}(f) \cong \pi_{r\omega_0+tA\omega_0}(f \circ (\sigma \otimes \sigma)).$$

So we know that

$$\mathrm{Tr}\pi_{(\sigma\otimes\sigma)^*(r\omega_0+tA\omega_0)}(f) = \mathrm{Tr}\pi_{r\omega_0+tA\omega_0}(f \circ (\sigma \otimes \sigma)).$$

Thus, relative to the same basis $\{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$, for every general functional $l = r\omega + tA\omega$

$$\begin{aligned}
\int_{\mathbf{R}} \mathrm{Tr}\pi_{r\omega+tA\omega}(f)dt &= \int_{\mathbf{R}} \mathrm{Tr}\pi_{(\sigma\otimes\sigma)^*(r\omega_0+tA\omega_0)}(f)dt \\
&= \int_{\mathbf{R}} \mathrm{Tr}\pi_{r\omega_0+tA\omega_0}(f\circ(\sigma\otimes\sigma))dt \\
&= \int_{\mathfrak{g}_1^*} \frac{1}{r}(f\circ(\sigma\otimes\sigma)\circ\exp)\hat{}(r\omega_0+\lambda)d\lambda \\
&= \int_{\mathfrak{g}_1^*} \frac{1}{r}(f\circ\exp\circ(\sigma\otimes\sigma))\hat{}(r\omega_0+\lambda)d\lambda \\
&= \int_{\mathfrak{g}_1^*} \frac{1}{r}(f\circ\exp)\hat{}(r\sigma^*\omega_0+\sigma^*\lambda)d\lambda \\
&= \int_{\mathfrak{g}_1^*} \frac{1}{r}(f\circ\exp)\hat{}(r\omega+\lambda)d\lambda
\end{aligned}$$

Note that the last equality holds because Lebesgue measure is rotation-invariant. So far we get

$$\int_{\mathbf{R}} \mathrm{Tr}\pi_{r\omega+tA\omega}(f)dt = \int_{\mathfrak{g}_1^*} \frac{1}{r}(f\circ\exp)\hat{}(r\omega+\lambda)d\lambda.$$

Since $\mathfrak{g}_2^*$ is regarded as isomorphic to $\mathbf{R}^3$,

$$dv(r,\omega) = r^2 sin\varphi dr d\varphi d\theta$$

is Lebesgue measure on $\mathfrak{g}_2^*$. Multiplying both sides of the above equation by r and integrating over $\mathfrak{g}_2^*$, we get

$$\begin{aligned}
&\int_{\mathfrak{g}_2^*}\int_{\mathbf{R}} r\mathrm{Tr}\pi_{r\omega+tA\omega}(f)dtdv(r,\omega) \\
=& \int_{\mathfrak{g}_2^*}\int_{\mathfrak{g}_1^*} (f\circ\exp)\hat{}(r\omega+\lambda)d\lambda dv(r,\omega) \\
=& \int_{\mathbf{R}^3}\int_{\mathfrak{g}_1^*} (f\circ\exp)\hat{}(\alpha_1 Z_1^*+\alpha_2 Z_2^*+\alpha_3 Z_3^*+\lambda)d\lambda d\alpha_1 d\alpha_2 d\alpha_3 \\
=& \int_{\mathfrak{g}_2^*}\int_{\mathfrak{g}_1^*} (f\circ\exp)\hat{}(l+\lambda)d\lambda dl
\end{aligned}$$

Hence for every $f \in \mathcal{S}(G)$, we have

$$f(e) = \int_{\mathfrak{g}_2^*}\int_{\mathfrak{g}_1^*} (f\circ\exp)\hat{}(l+\lambda)d\lambda dl = \int_{\mathfrak{g}_2^*}\int_{\mathbf{R}} r\mathrm{Tr}\pi_{r\omega+tA\omega}(f)dtdv(r,\omega).$$

We now would like to answer the question posed at the beginning: why do formulas (1.1) and (1.2) give the same result: f(e) ?

For the equation (1.1) we start with the Malcev basis $\{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$ to get $V_T \cap U = \{\alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^* + \beta_3 Y_3^* : \ \alpha_3 \neq 0\}$. Since

$$\{r\omega + tA\omega | r > 0, t \in \mathbf{R},\ \omega \in \mathfrak{g}_2^*,\ |\omega| = 1\}$$

is a cross-section for all orbits of $\mathfrak{g}^*$ that correspond to infinite-dimensional representations, for each $l \in V_T \cap U$, we need to know which element $r\omega + tA\omega$ from this cross-section such that $l \in \mathcal{O}_{r\omega+tA\omega}$.

Actually for each $l = \alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^* + \beta_3 Y_3^* \in V_T \cap U$, let

$$r := \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$$

and let $\sigma \in \mathrm{SO}(3)$ such that

$$\omega := \sigma^* Z_3^* = \tfrac{\alpha_1}{r} Z_1^* + \tfrac{\alpha_2}{r} Z_2^* + \tfrac{\alpha_3}{r} Z_3^*.$$

Then

$$A\omega = \tfrac{\alpha_1}{r} Y_1^* + \tfrac{\alpha_2}{r} Y_2^* + \tfrac{\alpha_3}{r} Y_3^*.$$

We want to find $t \in \mathbf{R}$ such that

$(\alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^*) + \beta_3 Y_3^* = l \in \mathcal{O}_{r\omega+tA\omega} = (r\omega) + tA\omega + (A\omega)^\perp$, i.e., $\beta_3 Y_3^* \in tA\omega + (A\omega)^\perp$.

Since the two-dimensional plane $tA\omega + (A\omega)^\perp$ in $\mathfrak{g}_1^*$ is obtained by pushing the plane $(A\omega)^\perp$ out along the direction $A\omega$, t is the component of the vector $\beta_3 Y_3^*$ on the direction $A\omega$. In other words, t is the value of inner product of vectors $\beta_3 Y_3^*$ and $A\omega$ in $\mathfrak{g}_1^*$, i.e., $t = \langle \beta_3 Y_3^*, \tfrac{\alpha_1}{r} Y_1^* + \tfrac{\alpha_2}{r} Y_2^* + \tfrac{\alpha_3}{r} Y_3^* \rangle = \tfrac{\alpha_3}{r}\beta_3$. It follows that

$\beta_3 = \frac{r}{\alpha_3}t$, and hence $d\beta_3 = \frac{r}{\alpha_3}dt$. Therefore the equation (1.1) turns out to be

$$\begin{aligned} f(e) &= \int_{\mathbf{R}^3}\int_{\mathbf{R}} |\alpha_3| \mathrm{Tr}\pi_{\alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^* + \beta_3 Y_3^*}(f) d\beta_3 (d\alpha_1 d\alpha_2 d\alpha_3) \\ &= \int_{\mathfrak{g}_2^*}\int_{\mathbf{R}} |\alpha_3| \mathrm{Tr}\pi_{r\omega + tA\omega}(f) |\frac{r}{\alpha_3}| dt dv(r,\omega) \\ &= \int_{\mathfrak{g}_2^*}\int_{\mathbf{R}} r \mathrm{Tr}\pi_{r\omega + tA\omega}(f) dt dv(r,\omega) \end{aligned}$$

If we start with the equation (1.2), a similar argument leads to the same formula. This answers the question at the start of this section.

Corwin & Greenleaf describe the formula

$$f(e) = \int_{\mathfrak{g}_2^*}\int_{\mathbf{R}} r \mathrm{Tr}\pi_{r\omega + tA\omega}(f) dt dv(r,\omega)$$

as 'more' invariant than the other two forms. But it is surely not canonical. It depends on choices of bases for $\mathfrak{g}$, and on the map A as well. Our goal is to redescribe Plancherel measure on a support expressed in terms of $\mathrm{Aut}^*(\mathfrak{g})$, and to do this in greater generality.

2. Parametrizing Orbits by Automorphisms

Theorem 2.1. Let $\mathfrak{g} = \mathcal{F}_{n,2}$, the free 2-step nilpotent Lie algebra on n generators, and let $\mathfrak{g}^*$ be its dual space. Let G be the corresponding simply connected nilpotent Lie group. Let $\mathfrak{g}^*_{Max}$ be the set of all functionals whose orbits have maximal dimension. Let $\mathcal{O}_{Max}$ be the set of all orbits of maximal dimension. Fix $l_0 \in \mathfrak{g}^*_{Max}$. Then there is a one-to-one correspondence between the quotient space $\mathrm{Ad}^*_G \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$ and the set $\mathcal{O}_{Max}$.

Proof. There are two cases depending upon the positive integer n.

Case 1. Suppose n is odd.
Let $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$: then $\dim(\mathfrak{g}_2) = \frac{n(n-1)}{2}$. Pick a Strong Malcev basis $\mathcal{B} = \{X_1, X_2, \cdots, X_m\}$ for $\mathfrak{g}$ through $\mathfrak{g}_2$ where $m = \frac{n(n-1)}{2} + n$. Let $\mathcal{B}^* = \{X_1^*, X_2^*, \cdots, X_m^*\}$ be its dual basis in $\mathfrak{g}^*$. Then $\mathcal{B}^*$ is a Jordan-Hölder basis for $\mathfrak{g}^*$. Let $V_j = \mathrm{Span}_{\mathbf{R}}\{X^*_{j+1}, \cdots, X^*_m\}$ and $V_m = \{0\}$. Since $\mathfrak{g}_2$ is the center and $\mathfrak{g}$ is two-step, coadjoint orbits in $\mathfrak{g}^*$ can not have dimension greater than n. And by Lemma 1.3.2. in [1], coadjoint orbits are always of even dimension. Since n is odd, the maximal dimension of orbits can not be greater than n-1. By definition generic orbits have maximal dimension in each quotient space $\mathfrak{g}^*/V_j$. In particular, each generic orbit has the maximal possible dimension in $\mathfrak{g}^*$, though the converse is false. It follows that each generic orbit has dimension $n-1$ since we can achieve the maximum possible dimension n-1 by assigning the basis element $X_{\frac{n(n-1)}{2}+1}$ to $\mathfrak{r}_l$ and giving l values on the center such that the non-degenerate bilinear form has a block-diagonal matrix with block $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Now for each l and $l' \in \mathfrak{g}^*_{Max}$, $\dim \mathcal{O}_l = n - 1 = \dim \mathcal{O}_{l'}$. It follows that $\dim \mathfrak{r}_l = \frac{n(n-1)}{2} + 1 = \dim \mathfrak{r}_{l'}$ since $\dim \mathcal{O}_l = \dim(G/R_l) = \dim \mathfrak{g} - \dim \mathfrak{r}_l$. Let A^l_n and$A^{l'}_n$ be picked such that $\mathfrak{r}_l = \mathfrak{g}_2 \oplus \mathbf{R}A^l_n$ and $\mathfrak{r}_{l'} = \mathfrak{g}_2 \oplus \mathbf{R}A^{l'}_n$. Let B_l and $B_{l'}$ be the non-degenerate skew-symmetric bilinear forms corresponding to l and l' living on $\mathfrak{g}/\mathfrak{r}_l$ and $\mathfrak{g}/\mathfrak{r}_{l'}$ respectively. By Theorem 6 of Chapter 10 in [5], there exists a basis $\{\overline{A^l_1}, \overline{A^l_2}, \cdots, \overline{A^l_{n-1}}\}$ for $\mathfrak{g}/\mathfrak{r}_l$ such that with respect to this basis the non-degenerate form B_l has the matrix $\mathrm{diag}(B_1, B_2, \cdots, B_{\frac{n-1}{2}})$ where each B_i is a 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and similarly for $B_{l'}$ using the basis $\{\overline{A^{l'}_1}, \overline{A^{l'}_2}, \cdots, \overline{A^{l'}_{n-1}}\}$.

Since l and $l' \in \mathfrak{g}^*_{Max}$, we know $l\mid_{\mathfrak{g}_2} \neq 0$ and $l\mid_{\mathfrak{g}_2} \neq 0$. Then let's subtract multiples of a central vector from each A^l_i and $A^{l'}_i$ to make $l(A^l_i - c_i Z^l) = l'(A^{l'}_i - c'_i Z^{l'})$, and still denote $A^l_i - c_i Z^l$ and $A^{l'}_i - c'_i Z^{l'}$ by A^l_i and $A^{l'}_i$. Thus we have

$$l(A^l_i) = l'(A^{l'}_i)$$

$\forall i = 1, 2, \cdots, n$.

Now let's define two bases for $\mathfrak{g}$ through $\mathfrak{g}_2$

$$\mathcal{B}^l = \{[A^l_i, A^l_j] : 1 \leq i < j \leq n\} \cup \{A^l_1, A^l_2, \cdots, A^l_n\}$$

and

$$\mathcal{B}^{l'} = \{[A^{l'}_i, A^{l'}_j] : 1 \leq i < j \leq n\} \cup \{A^{l'}_1, A^{l'}_2, \cdots, A^{l'}_n\}$$

Then define the mapping $\mathcal{A} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$A^l_i \longmapsto A^{l'}_i$$

and

$$[A^l_i, A^l_j] \longmapsto [A^{l'}_i, A^{l'}_j]$$

$1 \leq i < j \leq n$. It follows that $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ and $\mathcal{A}\mathfrak{r}_l = \mathfrak{r}_{l'}$. Hence we have the following identities

$$\begin{aligned}(\mathcal{A}^*l)([A_i^{l'}, A_j^{l'}]) &= l([\mathcal{A}^{-1}A_i^{l'}, \mathcal{A}^{-1}A_j^{l'}]) = l([A_i^l, A_j^l]) = B_l(\overline{A_i^l}, \overline{A_j^l}) = B_{l'}(\overline{A_i^{l'}}, \overline{A_j^{l'}}) \\ &= l'([A_i^{l'}, A_j^{l'}])\end{aligned}$$

$1 \leq i < j \leq n-1$. If j=n, since $\mathcal{A}_n^l \in \mathfrak{r}_l$ and $\mathcal{A}_n^{l'} \in \mathfrak{r}_{l'}$, we still have $(\mathcal{A}^*l)([A_i^{l'}, A_j^{l'}]) = l([A_i^l, A_j^l]) = 0 = l'([A_i^{l'}, A_j^{l'}])$ So $(\mathcal{A}^*l)|_{\mathfrak{g}_2} = l'|_{\mathfrak{g}_2}$.
And we know also that

$$(\mathcal{A}^*l)(A_i^{l'}) = l(\mathcal{A}^{-1}A_i^{l'}) = l(A_i^l) = l'(A_i^{l'})$$

$\forall i = 1, \cdots, n$. Thus we get the relation $\mathcal{A}^*l = l'$. Next we claim that $\mathcal{A}^*\mathcal{O}_l = \mathcal{O}_{l'}$. Here we abuse notation to denote by $\mathcal{A}$ the two corresponding automorphisms of $\mathfrak{g}$ and G respectively. For every $x \in G$, the map $i_x : g \longmapsto xgx^{-1}$ is an inner automorphism of G. Then we have

$$(\mathcal{A} \circ i_x)(g) = \mathcal{A}(xgx^{-1}) = (\mathcal{A}x)(\mathcal{A}g)(\mathcal{A}x)^{-1} = (i_{\mathcal{A}x} \circ \mathcal{A})(g)$$

$\forall g \in G$. So $\mathcal{A} \circ i_x = i_{\mathcal{A}x} \circ \mathcal{A}$. Then taking its differential at the unit element we get the identity $\mathcal{A} \circ \mathrm{Ad}_x = \mathrm{Ad}_{\mathcal{A}x} \circ \mathcal{A}$. By using the duality on both sides it follows that

$$\mathcal{A}^*\mathrm{Ad}_x^* = \mathrm{Ad}_{\mathcal{A}x}^*\mathcal{A}^*,$$

for all $x \in G$. Applying both maps to the functional l we get the relation $\mathcal{A}^*\mathcal{O}_l = \mathcal{O}_{\mathcal{A}^*l} = \mathcal{O}_{l'}$ as desired. In fact, the relation $\mathcal{A}^*\mathcal{O}_l = \mathcal{O}_{\mathcal{A}^*l}$ holds for all automorphisms $\mathcal{A}$.
Therefore for every $l, l' \in \mathfrak{g}_{Max}^*$, there exists an automorphism $\mathcal{A}$ of $\mathfrak{g}$ such that $\mathcal{A}^*l = l'$ and $\mathcal{A}^*\mathcal{O}_l = \mathcal{O}_{l'}$. Hence $\mathrm{Aut}^*(\mathfrak{g})$ maps $\mathfrak{g}_{Max}^*$ onto itself.

Case 2. Suppose n is even.

By arguments similar to those in the beginning of Case 1., since n is even, each generic orbit has dimension n: that is n is the maximal dimension for all orbits in $\mathfrak{g}^*$. Now for each $l, l' \in \mathfrak{g}^*_{Max}$, we have $\dim \mathfrak{r}_l = \dim \mathfrak{r}_{l'} = \frac{n(n-1)}{2} = \dim \mathfrak{g}_2$. Then $\mathfrak{r}_l = \mathfrak{g}_2 = \mathfrak{r}_{l'}$ since the center $\mathfrak{g}_2 \subseteq \mathfrak{r}_l$.

Now let B_l and $B_{l'}$ be the non-degenerate skew-symmetric bilinear forms corresponding to l and l' living on $\mathfrak{g}/\mathfrak{g}_2$ respectively. Then there exists a basis $\{\overline{A_1^l}, \overline{A_2^l}, \cdots, \overline{A_n^l}\}$ for $\mathfrak{g}/\mathfrak{g}_2$ giving a canonical matrix for B_l, and similarly for $B_{l'}$. Let's subtract multiples of a central vector from each A_i^l and $A_i^{l'}$ to make $l(A_i^l - c_i Z^l) = l'(A_i^{l'} - c_i' Z^{l'})$, and still denote $A_i^l - c_i Z^l$ and $A_i^{l'} - c_i' Z^{l'}$ by A_i^l and $A_i^{l'}$. Thus we have $l(A_i^l) = l'(A_i^{l'})$, $\forall i = 1, 2, \cdots, n$. Then define the map $\mathcal{A} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$A_i^l \longmapsto A_i^{l'}$$

and

$$[A_i^l, A_j^l] \longmapsto [A_i^{l'}, A_j^{l'}],$$

for $1 \le i < j \le n$. Then $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathcal{A}^* l = l'$ and $\mathcal{A}^* \mathcal{O}_l = \mathcal{O}_{l'}$ as before.

Thus, by cases 1 and 2 we know that $\mathrm{Aut}^*(\mathfrak{g})$ maps $\mathfrak{g}^*_{Max}$ onto itself.

By hypothesis $l_0 \in \mathfrak{g}^*_{Max}$, for every $\mathcal{A}, \mathcal{B} \in \mathrm{Aut}(\mathfrak{g})$ we know $\mathcal{A}^* l_0 = \mathcal{B}^* l_0 \Leftrightarrow \mathcal{B}^{*^{-1}} \mathcal{A}^* l_0 = l_0 \Leftrightarrow \mathcal{B}^{*^{-1}} \mathcal{A}^* \in \mathrm{Stab}(l_0) \Leftrightarrow \mathcal{A}^* \mathrm{Stab}(l_0) = \mathcal{B}^* \mathrm{Stab}(l_0)$.

So we obtain that $\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$ maps l_0 one-to-one and onto $\mathfrak{g}^*_{Max}$.

Now let's prove the mapping

$$\mathrm{Ad}_G^* \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0) \longrightarrow \mathcal{O}_{Max}$$

$$\mathrm{Ad}_G^* \mathcal{A}^* \mathrm{Stab}(l_0) \longmapsto \mathrm{Ad}_G^* \mathcal{A}^* l_0$$

is one-to-one.

For each double coset $\mathrm{Ad}_G^*\mathcal{A}^*\mathrm{Stab}(l_0)$ we get an orbit of maximal dimension, $\mathrm{Ad}_G^*\mathcal{A}^*l_0 \in \mathcal{O}_{Max}$. Suppose $\mathrm{Ad}_G^*\mathcal{A}^*l_0 = \mathrm{Ad}_G^*\mathcal{B}^*l_0$, then there exists a group element $g \in G$ such that

$$\mathrm{Ad}_g^*\mathcal{A}^*l_0 = \mathcal{B}^*l_0 \Leftrightarrow l_0 = \mathcal{A}^{*^{-1}}\mathrm{Ad}_{g^{-1}}^*\mathcal{B}^*l_0 \Leftrightarrow \mathcal{A}^{*^{-1}}\mathrm{Ad}_{g^{-1}}^*\mathcal{B}^* \in \mathrm{Stab}(l_0)$$
$$\Leftrightarrow \mathcal{B}^* \in \mathrm{Ad}_g^*\mathcal{A}^*\mathrm{Stab}(l_0).$$

So $\mathcal{B}^* \in \mathrm{Ad}_G^*\mathcal{A}^*\mathrm{Stab}(l_0)$, and hence $\mathcal{A}^*$ and $\mathcal{B}^*$ are in the same double coset. Therefore the mapping

$$\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0) \longrightarrow \mathcal{O}_{Max}$$

is one-to-one. This proves the theorem.

3. Relative Invariance of Plancherel Measure

Let S be a topological space and let G be a topological group acting on S. A measure μ on S is called a G-invariant measure if for every $g \in G$ and for every measurable subset E of S, the set gE is measurable and $\mu(gE) = \mu(E)$. It is very helpful if a non-trivial G-invariant Borel measure on a locally compact space S exists. But it is possible that no non-trivial invariant measure exists on S. This can happen already in the special case when G is a locally compact group, H is a closed subgroup of G and S is the homogeneous space G/H. A measure μ on a homogeneous space G/H is called *relatively invariant* provided $D(g)\mu(gE) \equiv \mu(E)\ \forall g \in G$ and $\forall E \subset G/H$ with E measurable. Then it is known that D is a homomorphism mapping $G \longrightarrow \mathbf{R}^+$ and that μ corresponds uniquely up to a constant factor to the homomorphism D. We call D the modular function for μ.

Theorems (3.1) and (3.2) are proven in Chapter 5 of [4].

Theorem (3.1) Let Δ be the modular function of the locally compact group G, and let δ be the modular function of the closed subgroup H. If there exists a relatively invariant measure μ on G/H, then its modular function D must be a continuous real character such that $D(h) = (\Delta/\delta)(h)$ for every $h \in H$.

Theorem (3.2) A necessary and sufficient condition for the existence of a non-trivial relatively invariant measure μ on G/H with modular function D is that D is a continuous real character on G such that $D(h) = (\Delta/\delta)(h)$ for every $h \in H$. If D satisfies the requirements, the relatively invariant measure associated with it is essentially uniquely determined by D.

Now we are ready to state and prove the main result of this chapter.

Theorem (3.3). Let $\mathfrak{g} = \mathcal{F}_{n,2}$. Fix $l_0 \in \mathfrak{g}^*_{Max}$, let μ be a copy of Plancherel measure on the double coset space $\mathrm{Ad}^*_G\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$. Then μ is the essentially unique relatively invariant measure corresponding to the homomorphism $\mathrm{Ad}^*_G\backslash\mathrm{Aut}^*(\mathfrak{g}) \longrightarrow \mathbf{R}^+$ given by $\overline{\mathcal{A}^*} \longmapsto \mid \det \overline{\mathcal{A}^*} \mid$. Note. $\det(\mathrm{Ad}^*_g) = 1$ for all $g \in G$, so $\det \overline{\mathcal{A}^*} \equiv \det \mathcal{A}^*$, $\forall \mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$.

Proof: For every automorphism $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, we want to show that

$$\mu(\overline{\mathcal{A}^*}E) = \mid \det \mathcal{A}^* \mid \mu(E)$$

for all measurable subsets E of $\mathrm{Ad}^*_G\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$.

For each $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ we also denote its corresponding automorphism of G by $\mathcal{A}$. For every $l \in \mathfrak{g}^*$ we claim that

$$\pi_{\mathcal{A}^{-1^*}l} \cong \pi_l \circ \mathcal{A}$$

For every $l \in \mathfrak{g}^*$, choose a polarizer $\mathcal{M}$ for l, and let $M = \exp(\mathcal{M})$. Then $\pi_l = \mathrm{Ind}^G_M \chi_{l,M}$ where $\chi_{l,M}$ is a character defined on M. It follows that

$$\begin{aligned} \pi_l \circ \mathcal{A} &= (\mathrm{Ind}^G_M \chi_{l,M}) \circ \mathcal{A} \\ &\cong \mathrm{Ind}^G_{\mathcal{A}^{-1}(M)}(\chi_{l,M} \circ \mathcal{A}) \end{aligned}$$

by Lemma 2.1.3. in [1]. For every $X \in \mathcal{M}$, since $(\chi_{l,M} \circ \mathcal{A})(\exp X) = \chi_{l,M}(\mathcal{A}(\exp X)) = \chi_{l,M}(\exp(\mathcal{A}X)) = e^{2\pi i l(\mathcal{A}X)} = e^{2\pi i(\mathcal{A}^{-1^*}l)(X)} = \chi_{\mathcal{A}^{-1^*}l,M}(\exp X)$, it follows further

$$\begin{aligned} \pi_l \circ \mathcal{A} &\cong \mathrm{Ind}^G_{\mathcal{A}^{-1}(M)} \chi_{\mathcal{A}^{-1^*}l,M} \\ &= \pi_{\mathcal{A}^{-1^*}l} \end{aligned}$$

This proves the claim. Then for every $\phi \in \mathcal{S}(G)$ we have

$$\begin{aligned}
\pi_{\mathcal{A}^{-1^*}l}(\phi) &= \int_G \phi(g)\pi_{\mathcal{A}^{-1^*}l}(g)dg \\
&\cong \int_G \phi(g)(\pi_l \circ \mathcal{A})(g)dg \\
&= \int_G \phi(g)\pi_l(\mathcal{A}g)dg \\
&= \int_G \phi(\mathcal{A}^{-1}x)\pi_l(x)d(\mathcal{A}^{-1}x) \\
&= \int_G (\phi \circ \mathcal{A}^{-1})(x)\pi_l(x)|\det \mathcal{A}^{-1}|dx \\
&= |\det \mathcal{A}^*| \int_G (\phi \circ \mathcal{A}^{-1})(x)\pi_l(x)dx \\
&= |\det \mathcal{A}^*|\pi_l(\phi \circ \mathcal{A}^{-1})
\end{aligned}$$

and hence

$$\mathrm{Tr}\pi_{\mathcal{A}^{-1^*}l}(\phi) = |\det \mathcal{A}^*|\mathrm{Tr}\pi_l(\phi \circ \mathcal{A}^{-1})$$

$\forall \mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, $l \in \mathfrak{g}^*$, $\phi \in \mathcal{S}(G)$.

Now let $D = \mathrm{Ad}_G^* \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$, note that the group $\mathrm{Ad}_G^* \backslash \mathrm{Aut}^*(\mathfrak{g})$ acts on the left on D. By hypothesis $l_0 \in \mathfrak{g}^*_{Max}$, we fix any strong Malcev basis $\mathcal{B}$ of $\mathfrak{g}$ such that $l_0 \in \mathfrak{g}^*_{gen}$ with respect to $\mathcal{B}$. Let $\mathcal{O}_{gen} = \{\mathcal{O}_l | \; l \in \mathfrak{g}^*_{gen}\}$. We know $\cup_{\mathcal{O} \in \mathcal{O}_{gen}} \mathcal{O}$ is a Zariski-open set in $\mathfrak{g}^*$, and Plancherel measure μ, as in [1], is supported on $\mathcal{O}_{gen} \subsetneq \mathcal{O}_{Max}$. Then define a copy of μ on $\mathcal{O}_{Max}$ by letting $\mu(\mathcal{O}_{Max} \backslash \mathcal{O}_{gen}) = 0$. By Theorem 2.1. we know there is a one-to-one correspondence between $\mathcal{O}_{Max}$ and D. So copy μ again onto D, in other words, μ is a copy of Plancherel measure on D. Let $E_0 = \{\overline{\mathcal{A}^*} \in D | \; \mathcal{A}^* l_0 \in \mathfrak{g}^*_{Max} \backslash \mathfrak{g}^*_{gen}\}$, then E_0 is a μ-nullset. For every measurable subset $E \subseteq D$, and every $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, we claim that

$$\mu(\overline{\mathcal{A}^*}E) = |\det \mathcal{A}^*|\mu(E)$$

$$
\begin{aligned}
\int_D \mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi)d\mu(\overline{\mathcal{A}^*}\,\dot{\mathcal{C}^*}) &= \int_D \mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi)d\mu((\mathcal{A}^*\dot{\mathcal{C}^*})) \\
&= \int_D \mathrm{Tr}\pi_{\mathcal{A}^{-1*}\mathcal{C}^* l_0}(\phi)d\mu(\dot{\mathcal{C}^*}) \\
&= \int_D |\det \mathcal{A}^*|\mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi\circ\mathcal{A}^{-1})d\mu(\dot{\mathcal{C}^*}) \\
&= |\det \mathcal{A}^*|\int_D \mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi\circ\mathcal{A}^{-1})d\mu(\dot{\mathcal{C}^*}) \\
&= |\det \mathcal{A}^*|(\phi\circ\mathcal{A}^{-1})(e) \\
&= |\det \mathcal{A}^*|\phi(e) \\
&= \int_D \mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi)|\det \mathcal{A}^*|d\mu(\dot{\mathcal{C}^*})
\end{aligned}
$$

$\forall\ \phi \in \mathcal{S}(G)$, where $\overline{\mathcal{A}^*}$ means coset $\mathrm{Ad}_G^*\mathcal{A}^* \in \mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})$ and $\dot{\mathcal{C}^*}$ means coset $\mathrm{Ad}_G^*\mathcal{C}^*\mathrm{Stab}(l_0) \in D$, and hence

$\overline{\mathcal{A}^*}\,\dot{\mathcal{C}^*} = \mathrm{Ad}_G^*\mathcal{A}^*\mathrm{Ad}_G^*\mathcal{C}^*\mathrm{Stab}(l_0) = \mathrm{Ad}_G^*(\mathcal{A}^*\mathcal{C}^*)\mathrm{Stab}(l_0) = (\mathcal{A}^*\dot{\mathcal{C}^*})$.

Thus we get the identity

$$\int_D \mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi)d\mu(\overline{\mathcal{A}^*}\,\dot{\mathcal{C}^*}) = \int_D \mathrm{Tr}\pi_{\mathcal{C}^* l_0}(\phi)\ |\det \mathcal{A}^*|\ d\mu(\dot{\mathcal{C}^*}).$$

Then by the Abstract Plancherel Theorem in Chapter 7 of [8], or in [2], one concludes, since μ is essentially unique, that

$$\mu(\overline{\mathcal{A}^*}E) = |\det \mathcal{A}^*|\ \mu(E)$$

for all $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ and all measurable subsets $E \subseteq D$. Therefore μ is the essentially unique relatively invariant measure on the double coset space $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$ corresponding to the homomorphism given by $\overline{\mathcal{A}^*} \longmapsto |\det \mathcal{A}^*| = |\det \overline{\mathcal{A}^*}|$.

Remark 1. This theorem can be generalized readily as follows. Let G be any group to which 'the Abstract Plancherel Theorem' applies. For every $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, define $\mathcal{A}^*\pi = \pi \circ \mathcal{A}^{-1}$, $\forall \pi \in \widehat{G}$. Then $\mathrm{Tr}((\mathcal{A}^*\pi)(\phi)) = |\det \mathcal{A}|\mathrm{Tr}(\pi(\phi \circ \mathcal{A}))$. Now suppose further that for every π in the support of Plancherel measure μ, the set $\{\mathcal{A}^*\pi|\ \mathcal{A} \in \mathrm{Aut}(G)\}$ contains the support of Plancherel measure. Let $\mathrm{Stab}(\pi) = \{\mathcal{A} \in \mathrm{Aut}(G)|\ \mathcal{A}^*\pi \cong \pi\}$. Then μ can be transported to a new domain $\mathrm{Aut}(G)/\mathrm{Stab}(\pi)$, where μ is 0 on the complement of its support. Then the same argument as at the top of page 15 shows that $\mu(\overline{\mathcal{A}^*}E) \equiv |\det \overline{\mathcal{A}^*}|\mu(E)$. Thus μ displays the same relative invariance whenever it can be transported to the domain $\mathrm{Aut}(G)/\mathrm{Stab}(\pi)$. In the case of $\mathcal{F}_{n,2}$, we have exploited the symmetry of the group to prove that $\mathrm{Aut}^*(G)\pi$ covers the support of the Plancherel measure whenever π is in that support.

Remark 2. Consider the example of the $2n+1$ dimensional Heisenberg group H_{2n+1}, which is not free if $n > 1$. Let $\mathfrak{h}_{2n+1}$ be its Heisenberg algebra which is spanned by $\{Z, Y_1, \cdots, Y_n, X_1, \cdots, X_n\}$ with non-trivial brackets $[X_i, Y_i] = Z$, $\forall i = 1, \cdots n$. Consider the generic representations π_l requiring $l(Z) \neq 0$. Let δ_r be the usual dilation automorphism, and δ_r^* be its dual, i.e.,

$$\delta_r : \left\{ \begin{array}{ll} X_i & \longmapsto \frac{1}{\sqrt{r}} X_i \\ Y_i & \longmapsto \frac{1}{\sqrt{r}} Y_i \\ Z & \longmapsto \frac{1}{r} Z \end{array} \right\} \quad \text{and} \quad \delta_r^* : \left\{ \begin{array}{ll} X_i^* & \longmapsto \sqrt{r} X_i^* \\ Y_i^* & \longmapsto \sqrt{r} Y_i^* \\ Z^* & \longmapsto r Z^* \end{array} \right\}$$

Then $\delta_r^*(\pi_l) \cong \pi_{\delta_r^* l}$ and $(\delta_r^* l)(Z) = r l(Z) \neq 0$. Therefore there exist sufficient automorphisms for the first remark to apply to all these (non-free!) groups as well.

Remark 3. Consider the example of 3-step Chain group G generated by X, Y_1, Y_2, Z where $[X, Y_1] = Y_2$ and $[X, Y_2] = Z$ generate all non-trivial brackets. This is neither 2-step nor free. In Example (3.1.12) of [1] it is shown that all the generic representations π_l are given by $l = zZ^* + y_1 Y_1^*$ with $z \neq 0$ and $y_1 \in \mathbf{R}$. Its radical $\mathfrak{r}_l = \mathrm{Span}_{\mathbf{R}}\{Z, Y_1\}$, hence its polarizer is three-dimensional since the dimension of polarizer is the average of dimensions of whole algebra and radical. Since $\mathfrak{m} = \mathrm{Span}_{\mathbf{R}}\{Z, Y_2, Y_1\}$ is an abelian ideal in $\mathfrak{g}$ of the correct dimension, it is a polarizer for each $l = zZ^* + y_1Y_1^*$.

Now consider the following automorphism $T_{A,e}$ indexed by

$A = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \in \mathrm{GL}(2, \mathbf{R})$ and $e \in \mathbf{R}$ given by

$$T_{A,e} : \left\{ \begin{array}{ll} X & \longmapsto a_1X \\ Y_1 & \longmapsto a_2X + a_3Y_1 + eZ \\ Y_2 & \longmapsto (\det A)Y_2 \\ Z & \longmapsto (a_1 \det A)Z \end{array} \right\}$$

By Kirillov theory we know each generic π_{z,y_1} is induced by a character χ_{z,y_1} defined on polarizer $\mathfrak{m}$. Then $\pi_{z,y_1} \circ T_{A,e} \cong \pi_{T_{A,e}^{-1*} l_{z,y_1}} = \pi_{(a_1 \det A)z, a_3y_1+ez}$. So any generic π can be carried into any other such representation by a suitable automorphism. Thus, Remark 1. applies to this case too.

4. Nonexistence of Invariant Measure

Proposition 4.1. Let $\mathfrak{g} = \mathcal{F}_{3,2}$. For $l_0 \in \mathfrak{g}^*_{Max}$, there is no $\overline{\mathrm{Aut}^*(\mathfrak{g})}$-invariant measure on the double coset space $\mathrm{Ad}^*_G \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$.

Proof. Pick a basis $\mathcal{B} = \{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$ for $\mathfrak{g}$ just as we did before in the Introduction, and let $\mathcal{B}^* = \{Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*\}$ be its dual basis in $\mathfrak{g}^*$. We start with the simple case in which $l_0 = Z_3^*$, then we will show that the result in the general case follows immediately. We begin by identifying $\mathrm{Stab}^*(l_0)$ explicitly. By the definition

$$\begin{aligned}
\mathrm{Stab}^*(l_0) &= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) |\ \mathcal{A}^* l_0 = l_0\} \\
&= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) |\ l_0 = \mathcal{A}^{-1^*} l_0\} \\
&= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) |\ l_0(X) = (\mathcal{A}^{-1^*} l_0)(X),\ \forall X \in \mathfrak{g}\} \\
&= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) |\ l_0(X) = l_0(\mathcal{A}X),\ \forall X \in \mathfrak{g}\} \\
&= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) |\ l_0((\mathcal{A} - I)(X)) = 0,\ \forall X \in \mathfrak{g}\} \\
&= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) |\ (\mathcal{A} - I) : \mathfrak{g} \to \ker(l_0)\}
\end{aligned}$$

This says $\mathrm{Stab}^*(l_0)$ is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ such that the operator after subtracting the identity from each element maps $\mathfrak{g}$ into the kernel of l_0. For $l_0 = Z_3^*$, its kernel $\ker(l_0) = \mathrm{Span}_{\mathbf{R}}\{Y_1, Y_2, Y_3, Z_1, Z_2\}$. Then for each $\mathcal{A} \in \mathrm{Stab}^*(l_0)$, we verify two properties:

$$\mathcal{A}Z_3 \in Z_3 + \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2\} \tag{4.1}$$

$$\mathcal{A}(\ker(l_0)) = \ker(l_0) \tag{4.2}$$

Since $\mathcal{A} \in \mathrm{Stab}^*(l_0)$, we have $(\mathcal{A} - I)(Z_3) \in \ker(l_0)$, so $\mathcal{A}Z_3 \in Z_3 + \ker(l_0)$. Since each automorphism of $\mathfrak{g}$ maps $\mathfrak{g}_2$ onto $\mathfrak{g}_2$, it follows $\mathcal{A}Z_3 \in (Z_3 + \ker(l_0)) \cap \mathfrak{g}_2 = Z_3 + \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2\}$. This proves property (4.1).

For each $K \in \ker(l_0)$ we have $l_0(\mathcal{A}K) = (\mathcal{A}^{-1^*}l_0)(K) = l_0(K) = 0$, so $\mathcal{A}(\ker(l_0)) \subseteq \ker(l_0)$. By hypothesis $\mathcal{A} \in \mathrm{Stab}^*(l_0)$, so is $\mathcal{A}^{-1}$. Using the above argument we get $\mathcal{A}^{-1}(\ker(l_0)) \subseteq \ker(l_0)$, and hence $\ker(l_0) \subseteq \mathcal{A}(\ker(l_0))$. Thus we obtain $\mathcal{A}(\ker(l_0)) = \ker(l_0)$ as desired.

For each $\mathcal{A} \in \mathrm{Stab}^*(l_0)$, since $\mathcal{A}$ is also an automorphism of $\mathfrak{g}$, we know that $\mathcal{A}$ maps $\mathfrak{g}_2$ onto $\mathfrak{g}_2$. This reason and the property $\mathcal{A}(\ker(l_0)) = \ker(l_0)$ together imply that

$$\mathcal{A}|_{\mathrm{Span}_{\mathbf{R}}\{Z_1,Z_2\}} : \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2\} \longrightarrow \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2\} \tag{4.3}$$

By properties (4.1), (4.2), and (4.3) we get a 6×6 matrix

$$[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p_1 & p_4 & p_7 \\ 0 & 0 & 0 & p_2 & p_5 & p_8 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right] =: \left[\begin{array}{c|c} Q_{3\times3} & R_{3\times3} \\ \hline 0_{3\times3} & P_{3\times3} \end{array}\right]$$

Then $\det \mathcal{A} = (\det Q)(\det P) = (\det P)^2(\det P) = (\det P)^3$ by Lemma 6.3. Since $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, $\mathcal{A}$ is invertible, and hence so is the matrix P. In other words, $\{\mathcal{A}Y_1, \mathcal{A}Y_2, \mathcal{A}Y_3\}$ is linearly independent modulo $\mathfrak{g}_2$. Since $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, this means $\mathcal{A}([Y_i, Y_j]) = [\mathcal{A}Y_i, \mathcal{A}Y_j]$. Explicitly,

$$\begin{aligned} \mathcal{A}(Z_1) &= \mathcal{A}([Y_2, Y_3]) = [\mathcal{A}Y_2, \mathcal{A}Y_3] = [p_4Y_1 + p_5Y_2 + p_6Y_3, p_7Y_1 + p_8Y_2 + p_9Y_3] \\ &= (p_5p_9 - p_6p_8)Z_1 - (p_4p_9 - p_6p_7)Z_2 + (p_4p_8 - p_5p_7)Z_3 \end{aligned}$$

$$\begin{aligned} \mathcal{A}(Z_2) &= \mathcal{A}([Y_3, Y_1]) = [\mathcal{A}Y_3, \mathcal{A}Y_1] = [p_7Y_1 + p_8Y_2 + p_9Y_3, p_1Y_1 + p_2Y_2 + p_3Y_3] \\ &= -(p_2p_9 - p_3p_8)Z_1 + (p_1p_9 - p_3p_7)Z_2 - (p_1p_8 - p_2p_7)Z_3 \end{aligned}$$

$$\begin{aligned}\mathcal{A}(Z_3) &= \mathcal{A}([Y_1, Y_2]) = [\mathcal{A}Y_1, \mathcal{A}Y_2] = [p_1Y_1 + p_2Y_2 + p_3Y_3, p_4Y_1 + p_5Y_2 + p_6Y_3] \\ &= (p_2p_6 - p_3p_5)Z_1 - (p_1p_6 - p_3p_4)Z_2 + (p_1p_5 - p_2p_4)Z_3\end{aligned}$$

Then we get the corresponding matrix

$$Q = \begin{bmatrix} \det\begin{pmatrix} p_5 & p_8 \\ p_6 & p_9 \end{pmatrix} & -\det\begin{pmatrix} p_2 & p_8 \\ p_3 & p_9 \end{pmatrix} & \det\begin{pmatrix} p_2 & p_5 \\ p_3 & p_6 \end{pmatrix} \\ -\det\begin{pmatrix} p_4 & p_7 \\ p_6 & p_9 \end{pmatrix} & \det\begin{pmatrix} p_1 & p_7 \\ p_3 & p_9 \end{pmatrix} & -\det\begin{pmatrix} p_1 & p_4 \\ p_3 & p_6 \end{pmatrix} \\ \det\begin{pmatrix} p_4 & p_7 \\ p_5 & p_8 \end{pmatrix} & -\det\begin{pmatrix} p_1 & p_7 \\ p_2 & p_8 \end{pmatrix} & \det\begin{pmatrix} p_1 & p_4 \\ p_2 & p_5 \end{pmatrix} \end{bmatrix} =: \begin{bmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{bmatrix}$$

Since the third row of Q is $[0, 0, 1]$, we obtain three conditions:

$$\det\begin{pmatrix} p_4 & p_7 \\ p_5 & p_8 \end{pmatrix} = 0, \ -\det\begin{pmatrix} p_1 & p_7 \\ p_2 & p_8 \end{pmatrix} = 0, \ \det\begin{pmatrix} p_1 & p_4 \\ p_2 & p_5 \end{pmatrix} = 1.$$

We analyze them by two cases.

Case 1. Suppose $p_1 \neq 0$.

Since we have the conditions

$$p_4p_8 - p_5p_7 = 0, \quad p_1p_8 - p_2p_7 = 0, \quad p_1p_5 - p_2p_4 = 1,$$

it follows that $p_8 = \frac{p_2p_7}{p_1}$ and $p_5 = \frac{1+p_2p_4}{p_1}$, and hence the first condition $0 = p_4p_8 - p_5p_7 = p_4\frac{p_2p_7}{p_1} - \frac{1+p_2p_4}{p_1}p_7 = -\frac{p_7}{p_1}$. Then $p_7 = 0$ and $p_8 = 0$. So we get the matrix

$$[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} q_1 & q_4 & q_7 & r_1 & r_4 & r_7 \\ q_2 & q_5 & q_8 & r_2 & r_5 & r_8 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p_1 & p_4 & 0 \\ 0 & 0 & 0 & p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

Since the matrix P is invertible, it follows $\det P = p_9 \neq 0$. Hence for each $\mathcal{A} \in \mathrm{Stab}^*(Z_3^*)$ with $p_1 \neq 0$, $\mathcal{A}$ has the above form of 6×6 matrix relative to the ordered basis $\mathcal{B}$ with the property $\det \mathcal{A} = (\det P)^3 = p_9^3 \neq 0$. Thus, the matrix Q is completely determined by matrix P, and matrices P and R have a total of 12 variables. Hence $\dim(\mathrm{Stab}^*(Z_3^*)) = 12$.

Case 2. Suppose $p_1 = 0$.

The second and third condition follow that $p_2p_7 = 0$ and $p_2p_4 = -1$. Then $p_2 \neq 0$, and hence $p_7 = 0$. So the first condition follows that $p_4p_8 = 0$, then $p_8 = 0$ since $p_4 = -\frac{1}{p_2} \neq 0$. Thus we get the matrix

$$[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} q_1 & q_4 & q_7 & r_1 & r_4 & r_7 \\ q_2 & q_5 & q_8 & r_2 & r_5 & r_8 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{p_2} & 0 \\ 0 & 0 & 0 & p_2 & p_5 & 0 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

with $\det \mathcal{A} = (\det P)^3 = p_9^3 \neq 0$. So the case $p_1 = 0$ corresponds to an 11-dimensional submanifold of the 12-dimensional manifold $\mathrm{Stab}^*(Z_3^*)$.

From the above two cases we see that we can always use p_1 as one of the coordinates in every local chart for $\mathrm{Stab}^*(Z_3^*)$. If $p_1 \neq 0$, we get the local coordinates given in the first case. If $p_1 = 0$, we get an 11-dimensional submanifold of $\mathrm{Stab}^*(Z_3^*)$ corresponding to $p_1 = 0$.

From Theorem 3.3. we know that the modular function D of the double coset space $\mathcal{D} := \mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$ is $D(\overline{\mathcal{A}^*}) = (\frac{\Delta}{\delta})(\overline{\mathcal{A}^*}) = |\det\overline{\mathcal{A}^*}|^{-1} = |\det\mathcal{A}|$ for every $\overline{\mathcal{A}^*} \in \mathrm{Ad}_G^*\backslash\mathrm{Ad}_G^*\cdot\mathrm{Stab}(l_0)$, i.e., $\mathcal{A} \in \mathrm{Stab}^*(l_0)$. And from the previous two cases, we already got $|\det\mathcal{A}| = |p_9^3|$, $\forall\mathcal{A} \in \mathrm{Stab}^*(l_0)$, which is not identically 1. Therefore there does not exist an invariant measure on $\mathcal{D}$ since the modular functions Δ and δ of $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})$ and $\mathrm{Ad}_G^*\backslash\mathrm{Ad}_G^*\cdot\mathrm{Stab}(l_0)$, respectively, do not agree on the subgroup $\mathrm{Ad}_G^*\backslash\mathrm{Ad}_G^*\cdot\mathrm{Stab}(l_0)$ for $l_0 = Z_3^*$. We next show the same result for general functionals $l_0 \in \mathfrak{g}_{Max}^*$.

For each $l_0 \in \mathfrak{g}_{Max}^*$, we know already that there exists an automorphism $\mathcal{A}_0$ of $\mathfrak{g}$ such that $\mathcal{A}_0^*Z_3^* = l_0$. Then it follows that

$$\begin{aligned}
\mathrm{Stab}^*(l_0) &= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})|\ \mathcal{A}^*l_0 = l_0\} \\
&= \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})|\ \mathcal{A}^*\mathcal{A}_0^*Z_3^* = \mathcal{A}_0^*Z_3^*\} \\
&= \{\mathcal{A}_0(\mathcal{A}_0^{-1}\mathcal{A}\mathcal{A}_0)\mathcal{A}_0^{-1}|\ (\mathcal{A}_0^{-1}\mathcal{A}\mathcal{A}_0)^*Z_3^* = Z_3^*\} \\
&= \mathcal{A}_0\{\mathcal{A}_0^{-1}\mathcal{A}\mathcal{A}_0|\ (\mathcal{A}_0^{-1}\mathcal{A}\mathcal{A}_0)^*Z_3^* = Z_3^*\}\mathcal{A}_0^{-1} \\
&= \mathcal{A}_0\{\mathcal{B} \in \mathrm{Aut}(\mathfrak{g})|\ \mathcal{B}^*Z_3^* = Z_3^*\}\mathcal{A}_0^{-1} \\
&= \mathcal{A}_0\mathrm{Stab}^*(Z_3^*)\mathcal{A}_0^{-1}
\end{aligned}$$

So for every $\mathcal{A} \in \mathrm{Stab}^*(l_0)$, $\mathcal{A} = \mathcal{A}_0\mathcal{C}\mathcal{A}_0^{-1}$ for some $\mathcal{C} \in \mathrm{Stab}^*(Z_3^*)$. And hence $|\det\mathcal{A}| = |\det(\mathcal{A}_0\mathcal{C}\mathcal{A}_0^{-1})| = |\det\mathcal{C}| = |p_9^3|$, which is not identically 1 either. Therefore we obtain the same conclusion that there is no invariant measure on the double coset space $\mathrm{Ad}^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$.

5. Explicit Calculations for $\mathcal{F}_{3,2}$

Brief Outline. There are seven lemmas and one verification in this long chapter for $\mathfrak{g} = \mathcal{F}_{3,2}$. We make a brief outline here.

By using the matrix format in the Proposition 4.1. , we find an almost global coordinate patch for $\mathrm{Ad}_G \backslash \mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$ in Lemma 5.1. In Lemma 5.2. we find a right and left Haar measure on $\mathrm{Ad}_G \backslash \mathrm{Ad}_G \cdot \mathrm{Stab}^*(l_0)$ with respect to the coordinate patch in Lemma 5.1. So its modular function is $\delta = \left|\frac{1}{p_9^5}\right| = \left|\frac{1}{(\det P)^5}\right|$ In Lemma 5.3. we find an almost global coordinate patch for $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})$. Then with respect to this patch, we figure out a right and left Haar measure on $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})$ in Lemma 5.4. Hence its modular function is $\Delta = \left|\frac{1}{(\det P)^2}\right|$

In Lemma 5.5. we construct automorphisms ρ_t^σ of $\mathfrak{g}$ related to automorphisms $\sigma \otimes \sigma$. Then we find a cross-section $\mathbf{X}$ for $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})/\mathrm{Stab}^*(l_0)$ in Lemma 5.6. Furthermore, in Lemma 5.7. we show to which element of $\mathbf{X}$ the composition of two elements of $\mathbf{X}$ corresponds since $\mathbf{X}$ is not a group. Then we use this concrete cross-section to verify the relative invariance for $\mathfrak{g} = \mathcal{F}_{3,2}$. This is the purpose of Chapter 5.

Throughout this chapter we have two ways to construct bases for $\mathcal{F}_{3,2}$. One involves norms, another does not. We build them here, and later just cite them for the hypothesis of each lemma.

Construction 1. Let $\mathfrak{g} = \mathcal{F}_{3,2}$, $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$, and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Choose a basis $\{\bar{Y}_1, \bar{Y}_2, \bar{Y}_3\}$ for $\mathfrak{g}/\mathfrak{g}_2$. Pick $Y_i \in \bar{Y}_i$, $\forall i = 1, 2, 3$. Let $Z_1 = [Y_2, Y_3]$, $Z_2 = [Y_3, Y_1]$, $Z_3 = [Y_1, Y_2]$, and these are independent of the choice of $Y_i \in \bar{Y}_i$. Then $\mathcal{B} := \{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$ is a strong Malcev basis of $\mathfrak{g}$ through $\mathfrak{g}_2$.

Let $\mathcal{B}^* := \{Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*\}$ be the dual basis of $\mathcal{B}$ in $\mathfrak{g}^*$, so $\mathcal{B}^*$ is a Jordan-Hölder basis of $\mathfrak{g}^*$. We know $\mathfrak{g}_2 = \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2, Z_3\}$, let $\mathfrak{g}_1 := \mathrm{Span}_{\mathbf{R}}\{Y_1, Y_2, Y_3\}$, $\mathfrak{g}_2^* := \mathrm{Span}_{\mathbf{R}}\{Z_1^*, Z_2^*, Z_3^*\}$, and $\mathfrak{g}_1^* := \mathrm{Span}_{\mathbf{R}}\{Y_1^*, Y_2^*, Y_3^*\}$.

Construction 2. Let $\mathfrak{g} = \mathcal{F}_{3,2}$, $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$, and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Give $\mathfrak{g}$ a norm respecting Lie bracket, and use the operator norm on $\mathfrak{g}^*$. Let $\mathfrak{g}_1$ be the orthogonal complement of $\mathfrak{g}_2$ in $\mathfrak{g}$. Choose an orthonormal basis $\{Y_1, Y_2, Y_3\}$ for $\mathfrak{g}_1$, then let $Z_1 = [Y_2, Y_3]$, $Z_2 = [Y_3, Y_1]$, and $Z_3 = [Y_1, Y_2]$. Since $\mathfrak{g}$ is equipped with a norm respecting bracket (which is defined at the first page of Chapter 6) , by Lemma 6.1. it follows that $\mathcal{B} := \{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$ is an orthonormal basis for $\mathfrak{g}$ through $\mathfrak{g}_2$. Let $\mathcal{B}^* := \{Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*\}$ be the dual basis of $\mathcal{B}$ in $\mathfrak{g}^*$, then $\mathcal{B}^*$ is also an orthonormal basis for $\mathfrak{g}^*$ by Lemma 6.2. We know $\mathfrak{g}_2 = \mathrm{Span}_{\mathbf{R}}\{Z_1, Z_2, Z_3\}$ and $\mathfrak{g}_1 = \mathrm{Span}_{\mathbf{R}}\{Y_1, Y_2, Y_3\}$, let $\mathfrak{g}_2^* := \mathrm{Span}_{\mathbf{R}}\{Z_1^*, Z_2^*, Z_3^*\}$ and $\mathfrak{g}_1^* := \mathrm{Span}_{\mathbf{R}}\{Y_1^*, Y_2^*, Y_3^*\}$.

Note. If the hypothesis does not require a norm of $\mathfrak{g}$ to respect the Lie bracket, then we can just build a basis $\mathcal{B}^*$ for $\mathfrak{g}^*$ by Construction 1., then give $\mathfrak{g}^*$ a Euclidean metric and corresponding norm by making $\mathcal{B}^*$ orthonormal.

Lemma 5.1. Let $\mathfrak{g} = \mathcal{F}_{3,2}$, and pick a basis $\mathcal{B}$ for $\mathfrak{g}$ just as we did in Construction 1. Let C_S be the set of all $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathcal{A}$ relative to the basis $\mathcal{B}$ has 6×6 matrix

$$\left[\begin{array}{ccc|ccc}
\frac{p_9}{p_1}(1+p_2p_4) & -p_2p_9 & p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4) & r_1 & r_4 & 0 \\
-p_4p_9 & p_1p_9 & p_3p_4 - p_1p_6 & 0 & r_5 & 0 \\
0 & 0 & 1 & r_3 & r_6 & 0 \\
\hline
0 & 0 & 0 & p_1 & p_4 & 0 \\
0 & 0 & 0 & p_2 & \frac{1+p_2p_4}{p_1} & 0 \\
0 & 0 & 0 & p_3 & p_6 & p_9
\end{array}\right]$$

with $p_i \in \mathbf{R}$, $p_1p_9 \neq 0$, $\forall i = 1,2,3,4,6,9$, and $r_j \in \mathbf{R}$, $\forall j = 1,3,4,5,6$. We claim that C_S is an almost global coordinate patch for $\mathrm{Ad}_G \backslash \mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$.

Proof. Since Ad_G is normal in $\mathrm{Aut}(\mathfrak{g})$, and $\mathrm{Stab}^*(Z_3^*)$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$, it follows their product $\mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$ is a group. we first need to re-parametrize cosets $\mathrm{Ad}_G s \in \mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$.

For every $X \in \mathfrak{g}$, let $X = \sum_{i=1}^3 z_i Z_i + \sum_{i=1}^3 y_i Y_i$. Since $\mathfrak{g}$ is two-step, we know $\mathrm{Ad}_{\exp(X)} = \exp(ad_X) = I + ad_X$. Since $ad_X Y_1 = [X, Y_1] = y_3 Z_2 - y_2 Z_3$, $ad_X Y_2 = [X, Y_2] = -y_3 Z_1 + Y_1 Z_3$, $ad_X Y_3 = [X, Y_3] = y_2 Z_1 - y_1 Z_2$, and $(ad_X)|_{\mathfrak{g}_2} = 0$, it follows

$$[\mathrm{Ad}_{\exp(X)}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -y_3 & y_2 \\ 0 & 1 & 0 & y_3 & 0 & -y_1 \\ 0 & 0 & 1 & -y_2 & y_1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] =: \left[\begin{array}{c|c} I_{3\times3} & S_{3\times3} \\ \hline 0_{3\times3} & I_{3\times3} \end{array}\right]$$

In Proposition 4.1. we have shown already that for almost all $s \in \mathrm{Stab}^*(Z_3^*)$

$$[s]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} \frac{p_9}{p_1}(1+p_2p_4) & -p_2p_9 & p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4) & r_1 & r_4 & r_7 \\ -p_4p_9 & p_1p_9 & p_3p_4 - p_1p_6 & r_2 & r_5 & r_8 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & p_1 & p_4 & 0 \\ 0 & 0 & 0 & p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

$$=: \left[\begin{array}{c|c} Q_{3\times3} & R_{3\times3} \\ \hline 0_{3\times3} & P_{3\times3} \end{array}\right]$$

Then

$$
\begin{aligned}
& \mathrm{Ad}_G s \\
= \ & \{\mathrm{Ad}_{\exp(X)} s \mid X = \sum_{i=1}^{3} z_i Z_i + \sum_{i=1}^{3} y_i Y_i, \ \forall z_i, y_i \in \mathbf{R}\} \\
= \ & \{\mathrm{Ad}_{\exp(X)} s \mid [\mathrm{Ad}_{\exp(X)} s]_B = \left[\begin{array}{c|c} I & S \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} Q & R \\ \hline 0 & P \end{array}\right] = \left[\begin{array}{c|c} Q & R+SP \\ \hline 0 & P \end{array}\right], \forall y_i \in \mathbf{R}\}
\end{aligned}
$$

where the 3×3 matrix

$$
\begin{aligned}
& R + SP \\
= \ & \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix} \\
= \ & \begin{bmatrix} r_1 + p_3y_2 - p_2y_3 & r_4 + p_6y_2 - \frac{1+p_2p_4}{p_1}y_3 & r_7 + p_9y_2 \\ r_2 - p_3y_1 + p_1y_3 & r_5 - p_6y_1 + p_4y_3 & r_8 - p_9y_1 \\ p_2y_1 - p_1y_2 & \frac{1+p_2p_4}{p_1}y_1 - p_4y_2 & 0 \end{bmatrix}
\end{aligned}
$$

Since the condition of $s \in \mathrm{Stab}^*(Z_3^*)$ is $p_1p_9 \neq 0$, we re-coordinate the matrix $R + SP$ by letting

$$
\begin{aligned}
s_1 &= r_2 - p_3y_1 + p_1y_3 \\
s_2 &= r_7 + p_9y_2 \\
s_3 &= r_8 - p_9y_1
\end{aligned}
$$

Then we have unique solution

$$
\begin{aligned}
y_1 &= (\ \frac{1}{p_9}r_8) - \frac{1}{p_9}s_3 \\
y_2 &= (\ -\frac{1}{p_9}r_7) + \frac{1}{p_9}s_2 \\
y_3 &= (-\frac{1}{p_9}r_2 + \frac{p_3}{p_1p_9}r_8) + \frac{1}{p_1}s_1 - \frac{p_3}{p_1p_9}s_3
\end{aligned}
$$

We get y_3 by means of y_1 and y_2 since

$s_1 = r_2 - p_3y_1 + p_1y_3 = r_2 - \frac{p_3}{p_9}r_8 + \frac{p_3}{p_9}s_3 + p_1y_3$, it follows

$p_1y_3 = (-r_2 + \frac{p_3}{p_9}r_8) + s_1 - \frac{p_3}{p_9}s_3$.

Let

$$\begin{aligned}\vec{P} &= \vec{P}(p_1, p_2, p_3, p_4, p_6, p_9)\\ \vec{R} &= \vec{R}(r_1, r_2, r_4, r_5, r_7, r_8)\\ \vec{S} &= \vec{S}(s_1, s_2, s_3)\end{aligned}$$

for each $P = \begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix}$, $R = \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ 0 & 0 & 0 \end{bmatrix}$, and triple (s_1, s_2, s_3).

Then we replace y_1, y_2, and y_3 by their corresponding identities in each entry of 3×3 matrix $R + SP$ to obtain

$$R + SP := \begin{bmatrix} R_1(\vec{P}, \vec{R}, \vec{S}) & R_4(\vec{P}, \vec{R}, \vec{S}) & s_2 \\ s_1 & R_5(\vec{P}, \vec{R}, \vec{S}) & s_3 \\ R_3(\vec{P}, \vec{R}, \vec{S}) & R_6(\vec{P}, \vec{R}, \vec{S}) & 0 \end{bmatrix}$$

Let's figure out the explicit formula for each $R_i(\vec{P}, \vec{R}, \vec{S})$, $i = 1, 3, 4, 5, 6$.

$$\begin{aligned}&R_1(\vec{P}, \vec{R}, \vec{S})\\ &= r_1 + p_3y_2 - p_2y_3\\ &= r_1 + p_3(-\tfrac{1}{p_9}r_7 + \tfrac{1}{p_9}s_2) - p_2(-\tfrac{1}{p_1}r_2 + \tfrac{p_3}{p_1p_9}r_8 + \tfrac{1}{p_1}s_1 - \tfrac{p_3}{p_1p_9}s_3)\\ &= (r_1 + \tfrac{p_2}{p_1}r_2 - \tfrac{p_3}{p_9}r_7 - \tfrac{p_2p_3}{p_1p_9}r_8) - \tfrac{p_2}{p_1}s_1 + \tfrac{p_3}{p_9}s_2 + \tfrac{p_2p_3}{p_1p_9}s_3,\end{aligned}$$

$R_3(\vec{P}, \vec{R}, \vec{S})$

$= p_2 Y_1 - p_1 y_2$

$= p_2(\frac{1}{p_9} r_8 - \frac{1}{p_9} s_3) - p_1(-\frac{1}{p_9} r_7 + \frac{1}{p_9} s_2)$

$= (\frac{p_1}{p_9} r_7 + \frac{p_2}{p_9} r_8) - \frac{p_1}{p_9} s_2 - \frac{p_2}{p_9} s_3,$

$R_4(\vec{P}, \vec{R}, \vec{S})$

$= r_4 + p_6 y_2 - \frac{1+p_2p_4}{p_1} y_3$

$= r_4 + p_6(-\frac{1}{p_9} r_7 + \frac{1}{p_9} s_2) - \frac{1+p_2p_4}{p_1}(-\frac{1}{p_1} r_2 + \frac{p_3}{p_1p_9} r_8 + \frac{1}{p_1} s_1 - \frac{p_3}{p_1p_9} s_3)$

$= (\frac{1+p_2p_4}{p_1^2} r_2 + r_4 - \frac{p_6}{p_9} r_7 - \frac{p_3(1+p_2p_4)}{p_1^2 p_9} r_8) - \frac{1+p_2p_4}{p_1^2} s_1 + \frac{p_6}{p_9} s_2 + \frac{p_3(1+p_2p_4)}{p_1^2 p_9} s_3,$

$R_5(\vec{P}, \vec{R}, \vec{S})$

$= r_5 - p_6 y_1 + p_4 y_3$

$= r_5 - p_6(\frac{1}{p_9} r_8 - \frac{1}{p_9} s_3) + p_4(-\frac{1}{p_1} r_2 + \frac{p_3}{p_1p_9} r_8 + \frac{1}{p_1} s_1 - \frac{p_3}{p_1p_9} s_3)$

$= (-\frac{p_4}{p_1} r_2 + r_5 - \frac{p_1p_6 - p_3p_4}{p_1p_9} r_8) + \frac{p_4}{p_1} s_1 + \frac{p_1p_6 - p_3p_4}{p_1p_9} s_3,$

$R_6(\vec{P}, \vec{R}, \vec{S})$

$= \frac{1+p_2p_4}{p_1} y_1 - p_4 y_2$

$= \frac{1+p_2p_4}{p_1}(\frac{1}{p_9} r_8 - \frac{1}{p_9} s_3) - p_4(-\frac{1}{p_9} r_7 + \frac{1}{p_9} s_2)$

$= (\frac{p_4}{p_9} r_7 + \frac{1+p_2p_4}{p_1p_9} r_8) - \frac{p_4}{p_9} s_2 - \frac{1+p_2p_4}{p_1p_9} s_3.$

Let $\vec{S} = \vec{0}$, i.e., $\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then we have unique solution

$$\begin{aligned} y_1 &= \frac{1}{p_9} r_8 \\ y_2 &= -\frac{1}{p_9} r_7 \\ y_3 &= -\frac{1}{p_1} r_2 + \frac{p_3}{p_1p_9} r_8 \end{aligned}$$

to make a unique automorphism $\mathrm{Ad}_{\exp(\sum_{i=1}^{3} y_i Y_i)} s$ in the coset $\mathrm{Ad}_G s$ with

$$[\mathrm{Ad}_{\exp(\sum_{i=1}^{3} y_i Y_i)} s]_{\mathcal{B}}$$

$$= \left[\begin{array}{ccc|ccc} \frac{p_9}{p_1}(1+p_2p_4) & -p_2p_9 & p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4) & R_1(\vec{P},\vec{R}) & R_4(\vec{P},\vec{R}) & 0 \\ -p_4p_9 & p_1p_9 & p_3p_4 - p_1p_6 & 0 & R_5(\vec{P},\vec{R}) & 0 \\ 0 & 0 & 1 & R_3(\vec{P},\vec{R}) & R_6(\vec{P},\vec{R}) & 0 \\ \hline 0 & 0 & 0 & p_1 & p_4 & 0 \\ 0 & 0 & 0 & p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

where each $R_j(\vec{P},\vec{R}) := R_j(\vec{P},\vec{R},\vec{0})$, $\forall j = 1,3,4,5,6$, i.e.,

$$\begin{aligned} R_1(\vec{P},\vec{R}) &= r_1 + \frac{p_2}{p_1}r_2 - \frac{p_3}{p_9}r_7 - \frac{p_2p_3}{p_1p_9}r_8 \\ R_3(\vec{P},\vec{R}) &= \frac{p_1}{p_9}r_7 + \frac{p_2}{p_9}r_8 \\ R_4(\vec{P},\vec{R}) &= \frac{1+p_2p_4}{p_1^2}r_2 + r_4 - \frac{p_6}{p_9}r_7 - \frac{p_3(1+p_2p_4)}{p_1^2p_9}r_8 \\ R_5(\vec{P},\vec{R}) &= -\frac{p_4}{p_1}r_2 + r_5 - \frac{p_1p_6 - p_3p_4}{p_1p_9}r_8 \\ R_6(\vec{P},\vec{R}) &= \frac{p_4}{p_9}r_7 + \frac{1+p_2p_4}{p_1p_9}r_8 \end{aligned}$$

Notice that $R_k(\vec{P},\vec{R},\vec{S}) = R_k(\vec{P},\vec{R}) + \{$ a linear function in s_i with rational coefficients in p_j $\}$

In summary, for each $s \in \mathrm{Stab}^*(Z_3^*)$ with $[s]_{\mathcal{B}} = \left[\begin{array}{c|c} Q_{3\times3} & R_{3\times3} \\ \hline 0_{3\times3} & P_{3\times3} \end{array}\right]$ such that $p_1p_9 \neq 0$ where $P = \begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix}$,

$$Q = \begin{bmatrix} \frac{p_9}{p_1}(1+p_2p_4) & -p_2p_9 & p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4) \\ -p_4p_9 & p_1p_9 & p_3p_4 - p_1p_6 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } R = \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ 0 & 0 & 0 \end{bmatrix},$$

we have the coset

$$
\begin{aligned}
& \mathrm{Ad}_G s \\
= & \ \{\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_i Y_i)} s \mid y_i \in \mathbf{R},\ i = 1,2,3\} \\
= & \left\{ \mathrm{Ad}_{\exp(\sum_{i=1}^3 y_i Y_i)} s \,\middle|\, \left\{ \begin{array}{rcl} y_1 & = & \frac{1}{p_9} r_8 - \frac{1}{p_9} s_3 \\ y_2 & = & -\frac{1}{p_9} r_7 + \frac{1}{p_9} s_2 \\ y_3 & = & -\frac{1}{p_1} r_2 + \frac{p_3}{p_1 p_9} r_8 + \frac{1}{p_1} s_1 - \frac{p_3}{p_1 p_9} s_3 \end{array} \right\} s_i \in \mathbf{R},\ \forall i = 1,2,3 \right\}
\end{aligned}
$$

with $[\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_i Y_i)} s]_B = \left[\begin{array}{c|c} Q_{3\times 3} & \tilde{R}_{3\times 3} \\ \hline 0_{3\times 3} & P_{3\times 3} \end{array}\right]$ where

$\tilde{R} = \begin{bmatrix} R_1(\vec{P},\vec{R},\vec{S}) & R_4(\vec{P},\vec{R},\vec{S}) & s_2 \\ s_1 & R_5(\vec{P},\vec{R},\vec{S}) & s_3 \\ R_3(\vec{P},\vec{R},\vec{S}) & R_6(\vec{P},\vec{R},\vec{S}) & 0 \end{bmatrix}$. Let $\vec{S} = \vec{0}$, then we have unique (y_1, y_2, y_3) to make unique $\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_i Y_i)} s \in \mathrm{Ad}_G s$ with

$[\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_i Y_i)} s]_B = \left[\begin{array}{c|c} Q_{3\times 3} & \tilde{R}^0_{3\times 3} \\ \hline 0_{3\times 3} & P_{3\times 3} \end{array}\right]$ where

$\tilde{R}^0 := \begin{bmatrix} R_1(\vec{P},\vec{R}) & R_4(\vec{P},\vec{R}) & 0 \\ 0 & R_5(\vec{P},\vec{R}) & 0 \\ R_3(\vec{P},\vec{R}) & R_6(\vec{P},\vec{R}) & 0 \end{bmatrix}$. Thus for each $s \in \mathrm{Stab}^*(Z_3^*)$, we get unique automorphism $\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_i Y_i)} s \in \mathrm{Ad}_G s$. Since each entry p_i of $P_{3\times 3}$ and r_j of $R_{3\times 3}$ varies in $\mathbf{R}$ with $p_1 p_9 \neq 0$, from the formula we know each $R_k(\vec{P},\vec{R})$ also varies in $\mathbf{R}$, $\forall i = 1,2,3,4,6,9$, $j = 1,2,4,5,7,8$, $k = 1,3,4,5,6$. Since each entry in $Q_{3\times 3}$ depends on p_i, let all p_i and r_j vary in $\mathbf{R}$ ($p_1 p_9 \neq 0$), we get almost the whole group $\mathrm{Stab}^*(Z_3^*)$ except a subset with measure zero. Since we already got unique automorphism from each coset $\mathrm{Ad}_G s$, which automorphism depends on each p_i

and R_k, let C_S be the collection of all those unique automorphisms. Explicitly,

$$C_S := \{\mathcal{A} \in \mathrm{Aut}(\mathfrak{g}) | p_i, R_k \in \mathbf{R} \ (p_1 p_9 \neq 0), \ \forall i = 1,2,3,4,6,9, \ \forall k = 1,3,4,5,6\}$$

with

$$[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} \frac{p_9}{p_1}(1+p_2p_4) & -p_2p_9 & p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4) & R_1 & R_4 & 0 \\ -p_4p_9 & p_1p_9 & p_3p_4 - p_1p_6 & 0 & R_5 & 0 \\ 0 & 0 & 1 & R_3 & R_6 & 0 \\ \hline 0 & 0 & 0 & p_1 & p_4 & 0 \\ 0 & 0 & 0 & p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

We next verify that different elements of C_S lie on different cosets. If this is done, then we obtain an almost global coordinate patch for $\mathrm{Ad}_G \backslash \mathrm{Ad}_G \cdot \mathrm{Stab}^{\bullet}(Z_3^{\bullet})$. For every distinct $\mathcal{A}, \mathcal{A}' \in C_S$, we know $\mathcal{A} \in \mathrm{Ad}_G s$ and $\mathcal{A}' \in \mathrm{Ad}_G s'$ for some $s, s' \in \mathrm{Stab}^{\bullet}(Z_3^{\bullet})$. Let $[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{c|c} Q_{3\times3} & \tilde{R}^0_{3\times3} \\ \hline 0_{3\times3} & P_{3\times3} \end{array}\right]$ and $[\mathcal{A}']_{\mathcal{B}} = \left[\begin{array}{c|c} Q'_{3\times3} & \tilde{R}'^0_{3\times3} \\ \hline 0_{3\times3} & P'_{3\times3} \end{array}\right]$ as above matrix. Then $\mathrm{Ad}_G s = \{\mathrm{Ad}_g s | [\mathrm{Ad}_g s]_{\mathcal{B}} = \left[\begin{array}{c|c} Q_{3\times3} & \tilde{R}_{3\times3} \\ \hline 0_{3\times3} & P_{3\times3} \end{array}\right], \forall \vec{S} \in \mathbf{R}^3\}$ and $\mathrm{Ad}_G s' = \{\mathrm{Ad}_g s' | [\mathrm{Ad}_g s']_{\mathcal{B}} = \left[\begin{array}{c|c} Q'_{3\times3} & \tilde{R}'_{3\times3} \\ \hline 0_{3\times3} & P'_{3\times3} \end{array}\right], \forall \vec{S} \in \mathbf{R}^3\}$. Since $\mathcal{A}$ and $\mathcal{A}'$ are distinct, it requires that at least one $p_i \neq p_i'$ or $R_j \neq R_j'$. If $p_i \neq p_i'$ for some $i \in \{1,2,3,4,6,9\}$, then $P_{3\times3} \neq P'_{3\times3}$, so $\mathrm{Ad}_G s \neq \mathrm{Ad}_G s'$. Suppose $p_i = p_i'$, $\forall i = 1,2,3,4,6,9$, i.e., $\vec{P} = \vec{P}'$. If $R_j \neq R_j'$ for some $j \in \{1,3,4,5,6\}$. For example, let $R_1 \neq R_1'$, i.e., $R_1(\vec{P},\vec{R},\vec{0}) \neq R_1'(\vec{P}',\vec{R}',\vec{0}) = R_1'(\vec{P},\vec{R}',\vec{0})$.

By formula of $R_1(\vec{P},\vec{R},\vec{S})$ we get

$$\begin{aligned} R_1(\vec{P},\vec{R},\vec{S}) &= R_1(\vec{P},\vec{R},\vec{0}) - \frac{p_2}{p_1}s_1 + \frac{p_3}{p_9}s_2 + \frac{p_2p_3}{p_1p_9}s_3 \\ &\neq R_1'(\vec{P},\vec{R}',\vec{0}) - \frac{p_2}{p_1}s_1 + \frac{p_3}{p_9}s_2 + \frac{p_2p_3}{p_1p_9}s_3 \\ &= R_1'(\vec{P},\vec{R}',\vec{S}) \\ &= R_1'(\vec{P}',\vec{R}',\vec{S}) \end{aligned}$$

So $R_1(\vec{P},\vec{R},\vec{S}) \neq R_1'(\vec{P}',\vec{R}',\vec{S})$, then $\bar{R}_{3\times3} \neq \bar{R}'_{3\times3}$, and hence $\mathrm{Ad}_G s \neq \mathrm{Ad}_G s'$. Similarly for other cases $R_i \neq R_i'$ with $i = 3,4,5,6$, we obtain the same result. Therefore different elements of C_S always lie on different cosets. One concludes that C_S is an almost global coordinate patch for $\mathrm{Ad}_G\backslash\mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$.

Lemma 5.2. The previous lemma says that C_S is an almost global coordinate patch for $\mathrm{Ad}_G\backslash\mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$. With respect to C_S we claim that

$$\frac{1}{p_9^6}dp_1dp_2dp_3dp_4dp_6dp_9dr_1dr_3dr_4dr_5dr_6$$

is a left Haar measure and

$$\frac{1}{p_9}dp_1dp_2dp_3dp_4dp_6dp_9dr_1dr_3dr_4dr_5dr_6$$

is a right Haar measure on $\mathrm{Ad}_G\backslash\mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$.

Proof. Given $\mathcal{A}_0 \in C_S$, for each $\mathcal{A} \in C_S$, let $\tilde{\mathcal{A}} = \mathcal{A}_0\mathcal{A}$. Since C_S is not a group, $\tilde{\mathcal{A}}$ may not be in C_S. We want to get the corresponding automorphism $\mathcal{A}'$ associated with $\tilde{\mathcal{A}}$ such that $\mathcal{A}' \in C_S$. Then we can calculate the determinant of the Jacobian of the map $\mathcal{A} \longmapsto \mathcal{A}'$ at the point $\mathcal{A}$ in order to figure out a left Haar measure on the quotient group $\mathrm{Ad}_G\backslash\mathrm{Ad}_G \cdot \mathrm{Stab}^*(Z_3^*)$.

Given $\mathcal{A}_0 \in C_S$, for each $\mathcal{A} \in C_S$, we know $\mathcal{A}_0 \in \mathrm{Ad}_G s_0$ and $\mathcal{A} \in \mathrm{Ad}_G s$ for some $s_0, s \in \mathrm{Stab}^*(Z_3^*)$. Then $\tilde{\mathcal{A}} \in (\mathrm{Ad}_G s_0)(\mathrm{Ad}_G s) = \mathrm{Ad}_G(s_0 s)$. Let

$$[\mathcal{A}_0]_{\mathcal{B}} = \left[\begin{array}{c|c} B_{3\times3} & C_{3\times3} \\ \hline 0_{3\times3} & A_{3\times3} \end{array}\right] \text{ and } [\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{c|c} Q_{3\times3} & R_{3\times3} \\ \hline 0_{3\times3} & P_{3\times3} \end{array}\right]$$

where $A = \begin{bmatrix} a_1 & a_4 & 0 \\ a_2 & \frac{1+a_2a_4}{a_1} & 0 \\ a_3 & a_6 & a_9 \end{bmatrix}$, $C = \begin{bmatrix} c_1 & c_4 & 0 \\ 0 & c_5 & 0 \\ c_3 & c_6 & 0 \end{bmatrix}$, $P = \begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix}$,

$R = \begin{bmatrix} r_1 & r_4 & 0 \\ 0 & r_5 & 0 \\ r_3 & r_6 & 0 \end{bmatrix}$, and B and Q are determined by A and P respectively, just like matrix formula in the collection C_S. Then

$$[\tilde{\mathcal{A}}]_{\mathcal{B}} = \left[\begin{array}{c|c} B & C \\ \hline 0 & A \end{array}\right] \left[\begin{array}{c|c} Q & R \\ \hline 0 & P \end{array}\right] = \left[\begin{array}{c|c} BQ & BR+CP \\ \hline 0 & AP \end{array}\right] =: \left[\begin{array}{c|c} Q'_{3\times3} & \tilde{R}_{3\times3} \\ \hline 0_{3\times3} & P'_{3\times3} \end{array}\right]$$

Since $\tilde{\mathcal{A}} \in \mathrm{Ad}_G(s_0 s)$ with $[\tilde{\mathcal{A}}]_{\mathcal{B}} = \left[\begin{array}{c|c} Q' & \tilde{R} \\ \hline 0 & P' \end{array}\right]$, let $[s_0 s]_{\mathcal{B}} = \left[\begin{array}{c|c} Q' & R' \\ \hline 0 & P' \end{array}\right]$

where R' is an unknown 3×3 matrix $\begin{bmatrix} r'_1 & r'_4 & r'_7 \\ r'_2 & r'_5 & r'_8 \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} &\mathrm{Ad}_G(s_0 s) \\ = \;& \left\{ \mathrm{Ad}_g(s_0 s) \,\middle|\, [\mathrm{Ad}_g(s_0 s)]_{\mathcal{B}} = \begin{bmatrix} Q'_{3\times3} & \begin{pmatrix} R_1(\vec{P}', \vec{R}', \vec{S}) & R_4(\vec{P}', \vec{R}', \vec{S}) & s_2 \\ s_1 & R_5(\vec{P}', \vec{R}', \vec{S}) & s_3 \\ R_3(\vec{P}', \vec{R}', \vec{S}) & R_6(\vec{P}', \vec{R}', \vec{S}) & 0 \end{pmatrix} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}, \forall \vec{S} \in \mathbf{R}^3 \right\} \end{aligned}$$

Thus, the corresponding automorphism $\mathcal{A}' \in \mathrm{Ad}_G(s_0 s)$ associated with $\tilde{\mathcal{A}}$ in the coordinate patch C_S has the matrix

$$[\mathcal{A}']_B = \begin{bmatrix} Q'_{3\times3} & \begin{pmatrix} R_1(\vec{P}', \vec{R}', \vec{0}) & R_4(\vec{P}', \vec{R}', \vec{0}) & 0 \\ 0 & R_5(\vec{P}', \vec{R}', \vec{0}) & 0 \\ R_3(\vec{P}', \vec{R}', \vec{0}) & R_6(\vec{P}', \vec{R}', \vec{0}) & 0 \end{pmatrix} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}$$

Note that we still have not known each $R_i(\vec{P}', \vec{R}', \vec{0})$ yet, since 3×3 matrix R' is unknown. But we will figure out later. First, let's calculate matrices P' and $\tilde{R}$.

$$\begin{aligned} & P' \\ = \; & A_{3\times3}P_{3\times3} \\ = \; & \begin{bmatrix} a_1 & a_4 & 0 \\ a_2 & \frac{1+a_2a_4}{a_1} & 0 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix} \\ = \; & \begin{bmatrix} a_1p_1 + a_4p_2 & a_1p_4 + a_4\frac{1+p_2p_4}{p_1} & 0 \\ a_2p_1 + \frac{1+a_2a_4}{a_1}p_2 & a_2p_4 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1} & 0 \\ a_3p_1 + a_6p_2 + a_9p_3 & a_3p_4 + a_6\frac{1+p_2p_4}{p_1} + a_9p_6 & a_9p_9 \end{bmatrix} \\ =: \; & \begin{bmatrix} p'_1 & p'_4 & 0 \\ p'_2 & \frac{1+p'_2p'_4}{p'_1} & 0 \\ p'_3 & p'_6 & p'_9 \end{bmatrix} \end{aligned}$$

For the entry $\frac{1+p_2'p_4'}{p_1'}$ of the last matrix, we need to check it equals $a_2p_4 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}$.

$$\begin{aligned}
&\frac{1+p_2'p_4'}{p_1'}\\
=\ &\frac{1+(a_2p_1+\frac{1+a_2a_4}{a_1}p_2)(a_1p_4+a_4\frac{1+p_2p_4}{p_1})}{a_1p_1+a_4p_2}\\
=\ &\frac{a_1a_2p_1p_4+a_2a_4p_2p_4}{a_1p_1+a_4p_2}+\frac{1+a_2a_4+(1+a_2a_4)p_2p_4+a_4p_2\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}\\
=\ &a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}
\end{aligned}$$

For the matrix $\tilde{R}$

$$\begin{aligned}
&\tilde{R}\\
=\ &BR+CP\\
=\ &\begin{bmatrix} \frac{a_9}{a_1}(1+a_2a_4) & -a_2a_9 & a_2a_6-\frac{a_3}{a_1}(1+a_2a_4) \\ -a_4a_9 & a_1a_9 & a_3a_4-a_1a_6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_1 & r_4 & 0 \\ 0 & r_5 & 0 \\ r_3 & r_6 & 0 \end{bmatrix}\\
&+\begin{bmatrix} c_1 & c_4 & 0 \\ 0 & c_5 & 0 \\ c_3 & c_6 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix}\\
=:\ &\begin{bmatrix} \tilde{r}_1' & \tilde{r}_4' & 0 \\ \tilde{r}_2' & \tilde{r}_5' & 0 \\ \tilde{r}_3' & \tilde{r}_6' & 0 \end{bmatrix}
\end{aligned}$$

where

$\tilde{r}_1' = \frac{a_9}{a_1}(1+a_2a_4)r_1 + (a_2a_6 - \frac{a_3}{a_1}(1+a_2a_4))r_3 + c_1p_1 + c_4p_2$

$\tilde{r}_2' = -a_4a_9r_1 + (a_3a_4 - a_1a_6)r_3 + c_5p_2$

$\tilde{r}_3' = r_3 + c_3p_1 + c_6p_2$

$\tilde{r}_4' = \frac{a_9}{a_1}(1+a_2a_4)r_4 - a_2a_9r_5 + (a_2a_6 - \frac{a_3}{a_1}(1+a_2a_4))r_6 + c_1p_4 + c_4\frac{1+p_2p_4}{p_1}$

$\tilde{r}_5' = -a_4a_9r_4 + a_1a_9r_5 + (a_3a_4 - a_1a_6)r_6 + c_5\frac{1+p_2p_4}{p_1}$

$\tilde{r}_6' = r_6 + c_3p_4 + c_6\frac{1+p_2p_4}{p_1}$

Since $\tilde{A} \in \mathrm{Ad}_G(s_0s)$ with $[\tilde{A}]_B = \begin{bmatrix} Q' & \tilde{R} \\ 0 & P' \end{bmatrix}$, it follows

$$\tilde{R} \in \left\{ \begin{bmatrix} R_1(\vec{P}',\vec{R}',\vec{S}) & R_4(\vec{P}',\vec{R}',\vec{S}) & s_2 \\ s_1 & R_5(\vec{P}',\vec{R}',\vec{S}) & s_3 \\ R_3(\vec{P}',\vec{R}',\vec{S}) & R_6(\vec{P}',\vec{R}',\vec{S}) & 0 \end{bmatrix} : \vec{S} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \in \mathbf{R}^3 \right\}$$

Thus if we let $\vec{S} = \begin{pmatrix} \tilde{r}_2' \\ 0 \\ 0 \end{pmatrix}$, we should get five equations: $\tilde{r}_k' = R_k(\vec{P}',\vec{R}',\vec{S})$, $k = 1,3,4,5,6$.

Next, let's try to get $R_k(\vec{P}',\vec{R}',\vec{0})$, and recall that

$R_k(\vec{P}',\vec{R}',\vec{S}) = R_k(\vec{P}',\vec{R}',\vec{0}) + \{\text{linear function in } s_i\}$, $\forall i = 1,2,3$.

Since

$\tilde{r}_1' = \frac{a_9}{a_1}(1+a_2a_4)r_1 + (a_2a_6 - \frac{a_3}{a_1}(1+a_2a_4))r_3 + c_1p_1 + c_4p_2$

$= R_1(\vec{P}',\vec{R}',\vec{S})$

$= R_1(\vec{P}',\vec{R}',\vec{0}) - \frac{p_2'}{p_1'}s_1 + \frac{p_3'}{p_9'}\cdot 0 + \frac{p_2'p_3'}{p_1'p_9'}\cdot 0,$

it follows

$$
\begin{aligned}
&R_1(\vec{P}',\vec{R}',\vec{0})\\
&= \tfrac{a_9}{a_1}(1+a_2a_4)r_1 + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4))r_3 + c_1p_1 + c_4p_2 + \tfrac{p_2'}{p_1'}s_1\\
&= \tfrac{a_9}{a_1}(1+a_2a_4)r_1 + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4))r_3 + c_1p_1 + c_4p_2\\
&\quad + \frac{a_2p_1+\frac{1+a_2a_4}{a_1}p_2}{a_1p_1+a_4p_2}(-a_4a_9r_1 + (a_3a_4 - a_1a_6)r_3 + c_5p_2)\\
&= (\tfrac{a_9}{a_1}(1+a_2a_4) - a_4a_9\frac{a_2p_1+\frac{1+a_2a_4}{a_1}p_2}{a_1p_1+a_4p_2})r_1\\
&\quad + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4) + (a_3a_4-a_1a_6)\frac{a_2p_1+\frac{1+a_2a_4}{a_1}p_2}{a_1p_1+a_4p_2})r_3 + \mathrm{rational}(\vec{P})\\
&= \frac{a_9p_1}{a_1p_1+a_4p_2}r_1 - \frac{a_3p_1+a_6p_2}{a_1p_1+a_4p_2}r_3 + \mathrm{rational}(\vec{P}).
\end{aligned}
$$

Since

$$
\tilde{r}_3' = r_3 + c_3p_1 + c_6p_2 = R_3(\vec{P}',\vec{R}',\vec{S}) = R_3(\vec{P}',\vec{R}',\vec{0}) - \tfrac{p_1'}{p_9'}\cdot 0 - \tfrac{p_2'}{p_9'}\cdot 0,
$$

it follows

$$
R_3(\vec{P}',\vec{R}',\vec{0}) = r_3 + c_3p_1 + c_6p_2 = r_3 + \mathrm{rational}(\vec{P}).
$$

Since

$$
\begin{aligned}
\tilde{r}_4' &= \tfrac{a_9}{a_1}(1+a_2a_4)r_4 - a_2a_9r_5 + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4))r_6 + c_1p_4 + c_4\tfrac{1+p_2p_4}{p_1}\\
&= R_4(\vec{P}',\vec{R}',\vec{S})\\
&= R_4(\vec{P}',\vec{R}',\vec{0}) - \tfrac{1+p_2'p_4'}{p_1'^2}s_1 + \tfrac{p_6'}{p_9'}\cdot 0 + \tfrac{p_3'(1+p_2'p_4')}{p_1'^2p_9'}\cdot 0
\end{aligned}
$$

and we know $\frac{1+p_2'p_4'}{p_1'} = a_2p_4 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}$,

it follows

$$
\begin{aligned}
&R_4(\vec{P}',\vec{R}',\vec{0})\\
&= \tfrac{a_9}{a_1}(1+a_2a_4)\mathrm{r}_4 - a_2a_9r_5 + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4))r_6 + c_1p_4 + c_4\tfrac{1+p_2p_4}{p_1} + \tfrac{1+p_2'p_4'}{p_1'^2}s_1\\
&= \tfrac{a_9}{a_1}(1+a_2a_4)r_4 - a_2a_9r_5 + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4))r_6 + c_1p_4 + c_4\tfrac{1+p_2p_4}{p_1}\\
&\quad + \frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}(-a_4a_9r_1 + (a_3a_4-a_1a_6)r_3 + c_5p_2)\\
&= -a_4a_9\frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}r_1 + (a_3a_4-a_1a_6)\frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}r_3\\
&\quad + \tfrac{a_9}{a_1}(1+a_2a_4)r_4 - a_2a_9r_5 + (a_2a_6 - \tfrac{a_3}{a_1}(1+a_2a_4))r_6 + \mathrm{rational}(\vec{P}).
\end{aligned}
$$

Since

$$\tilde{r}'_5 = -a_4a_9r_4 + a_1a_9r_5 + (a_3a_4 - a_1a_6)r_6 + c_5\tfrac{1+p_2p_4}{p_1}$$
$$= R_5(\vec{P}', \vec{R}, \vec{S})$$
$$= R_5(\vec{P}', \vec{R}', \vec{0}) + \tfrac{p'_4}{p'_1}s_1 + \tfrac{p'_1p'_6-p'_3p'_4}{p'_1p'_9}\cdot 0,$$

it follows

$$R_5(\vec{P}', \vec{R}', \vec{0})$$
$$= -a_4a_9r_4 + a_1a_9r_5 + (a_3a_4 - a_1a_6)r_6 + c_5\tfrac{1+p_2p_4}{p_1} - \tfrac{p'_4}{p'_1}s_1$$
$$= -a_4a_9r_4 + a_1a_9r_5 + (a_3a_4 - a_1a_6)r_6 + c_5\tfrac{1+p_2p_4}{p_1}$$
$$-\frac{a_1p_4+a_4\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}(-a_4a_9r_1 + (a_3a_4 - a_1a_6)r_3 + c_5p_2)$$
$$= a_4a_9\frac{a_1p_4+a_4\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}r_1 - (a_3a_4 - a_1a_6)\frac{a_1p_4+a_4\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}r_3$$
$$-a_4a_9r_4 + a_1a_9r_5 + (a_3a_4 - a_1a_6)r_6 + \text{rational}(\vec{P}).$$

Since

$$\tilde{r}'_6 = r_6 + c_3p_4 + c_6\tfrac{1+p_2p_4}{p_1}$$
$$= R_6(\vec{P}', \vec{R}', \vec{S})$$
$$= R_6(\vec{P}', \vec{R}', \vec{0}) - \tfrac{p'_4}{p'_9}\cdot 0 - \tfrac{1+p'_2p'_4}{p'_1p'_9}\cdot 0,$$

it follows

$$R_6(\vec{P}', \vec{R}', \vec{0})$$
$$= r_6 + c_3p_4 + c_6\tfrac{1+p_2p_4}{p_1}$$
$$= r_6 + \text{rational}(\vec{P}).$$

Thus we obtain all $R_k(\vec{P}', \vec{R}', \vec{0})$, and hence we get the corresponding $\mathcal{A}' \in \mathrm{Ad}_G(s_0s)$ associated with $\tilde{\mathcal{A}} = \mathcal{A}_0\mathcal{A}$ such that $\mathcal{A}' \in C_S$.

Then the determinant of the Jacobian of the map $\mathcal{A} \longmapsto \mathcal{A}'$ at point $\mathcal{A}$ is

$$\det\left[\frac{\partial(p_1', p_2', p_3', p_4', p_6', p_9', R_1, R_3, R_4, R_5, R_6)}{\partial(p_1, p_2, p_3, p_4, p_6, p_9, r_1, r_3, r_4, r_5, r_6)}\right] = \det\begin{bmatrix} \left(\frac{\partial p_i'}{\partial p_j}\right)_{6\times 6} & \left(\frac{\partial p_i'}{\partial r_j}\right)_{6\times 5} \\ \left(\frac{\partial R_i}{\partial p_j}\right)_{5\times 6} & \left(\frac{\partial R_i}{\partial r_j}\right)_{5\times 5} \end{bmatrix}$$

$$= \det\begin{bmatrix} \left(\frac{\partial p_i'}{\partial p_j}\right)_{6\times 6} & 0_{6\times 5} \\ \left(\frac{\partial R_i}{\partial p_j}\right)_{5\times 6} & \left(\frac{\partial R_i}{\partial r_j}\right)_{5\times 5} \end{bmatrix} = \det\left[\frac{\partial p_i'}{\partial p_j}\right]_{6\times 6} \det\left[\frac{\partial R_i}{\partial r_j}\right]_{5\times 5}$$

Let's calculate both determinants.

$$\begin{aligned}
& \det\left[\frac{\partial p_i'}{\partial p_j}\right]_{6\times 6} \\
= & \det\left[\frac{\partial(p_1', p_2', p_3', p_4', p_6', p_9')}{\partial(p_1, p_2, p_3, p_4, p_6, p_9)}\right] \\
= & \det\begin{bmatrix} a_1 & a_4 & 0 & 0 & 0 & 0 \\ a_2 & \frac{1+a_2a_4}{a_1} & 0 & 0 & 0 & 0 \\ a_3 & a_6 & a_9 & 0 & 0 & 0 \\ -a_4\frac{1+p_2p_4}{p_1^2} & a_4\frac{p_4}{p_1} & 0 & a_1 + a_4\frac{p_2}{p_1} & 0 & 0 \\ -a_6\frac{1+p_2p_4}{p_1^2} & a_6\frac{p_4}{p_1} & 0 & a_3 + a_6\frac{p_2}{p_1} & a_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_9 \end{bmatrix} \\
= & a_9 \cdot a_9 \cdot (a_1 + a_4\frac{p_2}{p_1}) \cdot a_9 \cdot \det\begin{bmatrix} a_1 & a_4 \\ a_2 & \frac{1+a_2a_4}{a_1} \end{bmatrix} \\
= & a_9^3 \frac{a_1p_1 + a_4p_2}{p_1} \\
= & (\det A)^3 \frac{p_1'}{p_1} \\
= & \frac{(\det P')^3 p_1'}{(\det P)^3 p_1}
\end{aligned}$$

$$
\begin{aligned}
&\det\left[\frac{\partial R_i}{\partial r_j}\right]_{5\times 5}\\
= \ &\det\left[\frac{\partial(R_1,R_3,R_4,R_5,R_6)}{\partial(r_1,r_3,r_4,r_5,r_6)}\right]\\
= \ &\det\begin{bmatrix}
\frac{a_9p_1}{a_1p_1+a_4p_2} & * & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
-a_4a_9\frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2} & * & \frac{a_9}{a_1}(1+a_2a_4) & -a_2a_9 & a_2a_6-\frac{a_3}{a_1}(1+a_2a_4)\\
a_4a_9\frac{a_1p_4+a_4\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2} & * & -a_4a_9 & a_1a_9 & a_3a_4-a_1a_6\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}\\
= \ &\det\begin{bmatrix}
\frac{a_9p_1}{a_1p_1+a_4p_2} & 0 & 0\\
-a_4a_9\frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2} & \frac{a_9}{a_1}(1+a_2a_4) & -a_2a_9\\
a_4a_9\frac{a_1p_4+a_4\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2} & -a_4a_9 & a_1a_9
\end{bmatrix}\\
= \ &\frac{a_9p_1}{a_1p_1+a_4p_2}(a_9^2(1+a_2a_4)-a_2a_4a_9^2)\\
= \ &a_9^3\frac{p_1}{a_1p_1+a_4p_2}\\
= \ &(\det A)^3\frac{p_1}{p_1'}\\
= \ &\frac{(\det P')^3p_1}{(\det P)^3p_1'}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\det\left[\frac{\partial(p_1',p_2',p_3',p_4',p_6',p_9',R_1,R_3,R_4,R_5,R_6)}{\partial(p_1,p_2,p_3,p_4,p_6,p_9,r_1,r_3,r_4,r_5,r_6)}\right]=\det\left[\frac{\partial p_i'}{\partial p_j}\right]_{6\times 6}\det\left[\frac{\partial R_i}{\partial r_j}\right]_{5\times 5}\\
= \ &\frac{(\det P')^3p_1'}{(\det P)^3p_1}\frac{(\det P')^3p_1}{(\det P)^3p_1'}=\frac{(\det P')^6}{(\det P)^6}=\frac{p_9'^6}{p_9^6}
\end{aligned}
$$

By the explanation at page 251. in [4], one concludes that

$$\tfrac{1}{p_9^6}dp_1dp_2dp_3dp_4dp_6dp_9dr_1dr_3dr_4dr_5dr_6$$

is a left Haar measure on $\mathrm{Ad}_G\backslash\mathrm{Ad}_G\cdot\mathrm{Stab}^*(Z_3^*)$. Next step we would like to find a right Haar measure with respect to the same patch C_S.

Given $\mathcal{A}_0 \in C_S$, for each $\mathcal{A} \in C_S$, let $\tilde{\mathcal{A}} = \mathcal{A}\mathcal{A}_0$. But $\tilde{\mathcal{A}}$ may not be in C_S, we want to get the corresponding $\mathcal{A}'$ associated with $\tilde{\mathcal{A}}$ such that $\mathcal{A}' \in C_S$. Since $\mathcal{A} \in \mathrm{Ad}_G s$ and $\mathcal{A}_0 \in \mathrm{Ad}_G s_0$ for some $s, s_0 \in \mathrm{Stab}^*(Z_3^*)$, we have $\tilde{\mathcal{A}} \in \mathrm{Ad}_G(ss_0)$.

Let $[\mathcal{A}]_{\mathcal{B}} = \begin{bmatrix} Q_{3\times3} & R_{3\times3} \\ 0_{3\times3} & P_{3\times3} \end{bmatrix}$ and $[\mathcal{A}_0]_{\mathcal{B}} = \begin{bmatrix} B_{3\times3} & C_{3\times3} \\ 0_{3\times3} & A_{3\times3} \end{bmatrix}$ as before. Then

$[\tilde{\mathcal{A}}]_{\mathcal{B}} = [\mathcal{A}\mathcal{A}_0]_{\mathcal{B}} = \begin{bmatrix} Q & R \\ 0 & P \end{bmatrix} \begin{bmatrix} B & C \\ 0 & A \end{bmatrix} = \begin{bmatrix} QB & QC+RA \\ 0 & PA \end{bmatrix} =:$

$\begin{bmatrix} Q'_{3\times3} & \tilde{R}_{3\times3} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}$,

and hence we can let $[ss_0]_{\mathcal{B}} = \begin{bmatrix} Q'_{3\times3} & R'_{3\times3} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}$ where $R'_{3\times3}$ is an unknown matrix

with the form $\begin{bmatrix} r'_1 & r'_4 & r'_7 \\ r'_2 & r'_5 & r'_8 \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned}
&\mathrm{Ad}_G(ss_0) \\
&= \{\mathrm{Ad}_g(ss_0) | [\mathrm{Ad}_g(ss_0)]_{\mathcal{B}} = \\
&\begin{bmatrix} Q'_{3\times3} & \begin{pmatrix} R_1(\vec{P}',\vec{R}',\vec{S}) & R_4(\vec{P}',\vec{R}',\vec{S}) & s_2 \\ s_1 & R_5(\vec{P}',\vec{R}',\vec{S}) & s_3 \\ R_3(\vec{P}',\vec{R}',\vec{S}) & R_6(\vec{P}',\vec{R}',\vec{S}) & 0 \end{pmatrix} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}, \vec{S} \in \mathbf{R}^3\}
\end{aligned}$$

And the corresponding $\mathcal{A}' \in C_S$ associated with $\tilde{\mathcal{A}}$ has the matrix

$$[\mathcal{A}']_{\mathcal{B}} = \begin{bmatrix} Q'_{3\times3} & \begin{pmatrix} R_1(\vec{P}',\vec{R}',\vec{0}) & R_4(\vec{P}',\vec{R}',\vec{0}) & 0 \\ 0 & R_5(\vec{P}',\vec{R}',\vec{0}) & 0 \\ R_3(\vec{P}',\vec{R}',\vec{0}) & R_6(\vec{P}',\vec{R}',\vec{0}) & 0 \end{pmatrix} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}$$

Notice that we still have not known each $R_k(\vec{P}', \vec{R}', \vec{0})$ yet, but we will figure out later. First, let's calculate matrices P' and $\tilde{R}$.

$$\begin{aligned}
& P' \\
= \; & PA \\
= \; & \begin{bmatrix} p_1 & p_4 & 0 \\ p_2 & \frac{1+p_2p_4}{p_1} & 0 \\ p_3 & p_6 & p_9 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & 0 \\ a_2 & \frac{1+a_2a_4}{a_1} & 0 \\ a_3 & a_6 & a_9 \end{bmatrix} \\
= \; & \begin{bmatrix} a_1p_1 + a_2p_4 & a_4p_1 + \frac{1+a_2a_4}{a_1}p_4 & 0 \\ a_1p_2 + a_2\frac{1+p_2p_4}{p_1} & a_4p_2 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1} & 0 \\ a_1p_3 + a_2p_6 + a_3p_9 & a_4p_3 + \frac{1+a_2a_4}{a_1}p_6 + a_6p_9 & a_9p_9 \end{bmatrix} \\
=: \; & \begin{bmatrix} p_1' & p_4' & 0 \\ p_2' & \frac{1+p_2'p_4'}{p_1'} & 0 \\ p_3' & p_6' & p_9' \end{bmatrix}
\end{aligned}$$

We need to check the entry $\frac{1+p_2'p_4'}{p_1'}$ of the last matrix equal to $a_4p_2 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}$.

$$\begin{aligned}
& \frac{1+p_2'p_4'}{p_1'} \\
= \; & \frac{1 + (a_1p_2 + a_2\frac{1+p_2p_4}{p_1})(a_4p_1 + \frac{1+a_2a_4}{a_1}p_4)}{a_1p_1 + a_2p_4} \\
= \; & \frac{1 + a_1a_4p_1p_2 + a_2a_4(1+p_2p_4) + (1+a_2a_4)p_2p_4 + a_2\frac{1+a_2a_4}{a_1}p_4\frac{1+p_2p_4}{p_1}}{a_1p_1 + a_2p_4} \\
= \; & \frac{a_4p_2(a_1p_1 + a_2p_4) + (1+a_2a_4)(1+p_2p_4) + a_2p_4\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1 + a_2p_4} \\
= \; & a_4p_2 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}
\end{aligned}$$

For the 3×3 matrix $\tilde{R}$

$$
\begin{aligned}
&\tilde{R} \\
= \; & QC + RA \\
= \; & \begin{bmatrix} \frac{p_9}{p_1}(1+p_2p_4) & -p_2p_9 & p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4) \\ -p_4p_9 & p_1p_9 & p_3p_4 - p_1p_6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & c_4 & 0 \\ 0 & c_5 & 0 \\ c_3 & c_6 & 0 \end{bmatrix} \\
& + \begin{bmatrix} r_1 & r_4 & 0 \\ 0 & r_5 & 0 \\ r_3 & r_6 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & 0 \\ a_2 & \frac{1+a_2a_4}{a_1} & 0 \\ a_3 & a_6 & a_9 \end{bmatrix} \\
= : \; & \begin{bmatrix} \tilde{r}'_1 & \tilde{r}'_4 & 0 \\ \tilde{r}'_2 & \tilde{r}'_5 & 0 \\ \tilde{r}'_3 & \tilde{r}'_6 & 0 \end{bmatrix}
\end{aligned}
$$

where

$\tilde{r}'_1 = a_1r_1 + a_2r_4 + c_1\frac{p_9}{p_1}(1+p_2p_4) + c_3(p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4))$

$\tilde{r}'_2 = a_2r_5 - c_1p_4p_9 + c_3(p_3p_4 - p_1p_6)$

$\tilde{r}'_3 = a_1r_3 + a_2r_6 + c_3$

$\tilde{r}'_4 = a_4r_1 + \frac{1+a_2a_4}{a_1}r_4 + c_4\frac{p_9}{p_1}(1+p_2p_4) - c_5p_2p_9 + c_6(p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4))$

$\tilde{r}'_5 = \frac{1+a_2a_4}{a_1}r_5 - c_4p_4p_9 + c_5p_1p_9 + c_6(p_3p_4 - p_1p_6)$

$\tilde{r}'_6 = a_4r_3 + \frac{1+a_2a_4}{a_1}r_6 + c_6$

Since $\tilde{\mathcal{A}} \in \mathrm{Ad}_G(ss_0)$ with $[\tilde{\mathcal{A}}]_{\mathcal{B}} = \begin{bmatrix} Q' & \tilde{R} \\ 0 & P' \end{bmatrix}$, it follows

$$\tilde{R} \in \left\{ \begin{bmatrix} R_1(\vec{P}', \vec{R}', \vec{S}) & R_4(\vec{P}', \vec{R}', \vec{S}) & s_2 \\ s_1 & R_5(\vec{P}', \vec{R}', \vec{S}) & s_3 \\ R_3(\vec{P}', \vec{R}', \vec{S}) & R_6(\vec{P}', \vec{R}', \vec{S}) & 0 \end{bmatrix} : \vec{S} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \in \mathbf{R}^3 \right\}$$

Thus if we let $\vec{S} = \begin{pmatrix} \tilde{r}'_2 \\ 0 \\ 0 \end{pmatrix}$, we should get five equations: $\tilde{r}'_k = R_k(\vec{P}', \vec{R}', \vec{S})$, $k = 1, 3, 4, 5, 6$. From these equations we can get $R_k(\vec{P}', \vec{R}', \vec{0})$.

Since

$$\begin{aligned}\tilde{r}'_1 &= a_1 r_1 + a_2 r_4 + c_1 \tfrac{p_9}{p_1}(1 + p_2 p_4) + c_3(p_2 p_6 - \tfrac{p_3}{p_1}(1 + p_2 p_4)) \\ &= R_1(\vec{P}', \vec{R}', \vec{S}) \\ &= R_1(\vec{P}', \vec{R}', \vec{0}) - \tfrac{p'_2}{p'_1} s_1 + \tfrac{p'_3}{p'_9} \cdot 0 + \tfrac{p'_2 p'_3}{p'_1 p'_9} \cdot 0,\end{aligned}$$

it follows

$$\begin{aligned}&R_1(\vec{P}', \vec{R}', \vec{0}) \\ &= a_1 r_1 + a_2 r_4 + c_1 \tfrac{p_9}{p_1}(1 + p_2 p_4) + c_3(p_2 p_6 - \tfrac{p_3}{p_1}(1 + p_2 p_4)) \\ &\quad + \tfrac{a_1 p_2 + a_2 \frac{1 + p_2 p_4}{p_1}}{a_1 p_1 + a_2 p_4}(a_2 r_5 - c_1 p_4 p_9 + c_3(p_3 p_4 - p_1 p_6)) \\ &= a_1 r_1 + a_2 r_4 + \tfrac{a_1 a_2 p_1 p_2 + a_2^2(1 + p_2 p_4)}{p_1(a_1 p_1 + a_2 p_4)} r_5 + \text{rational}(\vec{P}).\end{aligned}$$

Since

$$\begin{aligned}\tilde{r}'_3 &= a_1 r_3 + a_2 r_6 + c_3 \\ &= R_3(\vec{P}', \vec{R}', \vec{S}) \\ &= R_3(\vec{P}', \vec{R}', \vec{0}) - \tfrac{p'_1}{p'_9} \cdot 0 - \tfrac{p'_2}{p'_9} \cdot 0,\end{aligned}$$

it follows

$$\begin{aligned}&R_3(\vec{P}', \vec{R}', \vec{0}) \\ &= a_1 r_3 + a_2 r_6 + c_3.\end{aligned}$$

Since

$$\begin{aligned}\tilde{r}'_4 &= a_4 r_1 + \tfrac{1 + a_2 a_4}{a_1} r_4 + c_4 \tfrac{p_9}{p_1}(1 + p_2 p_4) - c_5 p_2 p_9 + c_6(p_2 p_6 - \tfrac{p_3}{p_1}(1 + p_2 p_4)) \\ &= \mathrm{R}_4(\vec{P}', \vec{R}', \vec{S}) \\ &= R_4(\vec{P}', \vec{R}', \vec{0}) - \tfrac{1 + p'_2 p'_4}{p'^2_1} s_1 + \tfrac{p'_6}{p'_9} \cdot 0 + \tfrac{p'_3(1 + p'_2 p'_4)}{p'^2_1 p'_9} \cdot 0\end{aligned}$$

and we know $\frac{1+p'_2p'_4}{p'_1} = a_4p_2 + \frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}$,

it follows

$$R_4(\vec{P}', \vec{R}', \vec{0})$$

$$= a_4r_1 + \frac{1+a_2a_4}{a_1}r_4 + c_4\frac{p_9}{p_1}(1+p_2p_4) - c_5p_2p_9 + c_6(p_2p_6 - \frac{p_3}{p_1}(1+p_2p_4))$$
$$+\frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}(a_2r_5 - c_1p_4p_9 + c_3(p_3p_4 - p_1p_6))$$
$$= a_4r_1 + \frac{1+a_2a_4}{a_1}r_4 + a_2\frac{a_2p_4+\frac{1+a_2a_4}{a_1}\frac{1+p_2p_4}{p_1}}{a_1p_1+a_4p_2}r_5 + \text{rational}(\vec{P})$$

Since

$$\vec{r}'_5 = \frac{1+a_2a_4}{a_1}r_5 - c_4p_4p_9 + c_5p_1p_9 + c_6(p_3p_4 - p_1p_6)$$
$$= R_5(\vec{P}', \vec{R}', \vec{S})$$
$$= R_5(\vec{P}', \vec{R}', \vec{0}) + \frac{p'_4}{p'_1}s_1 + \frac{p'_1p'_6 - p'_3p'_4}{p'_1p'_9} \cdot 0,$$

it follows

$$R_5(\vec{P}', \vec{R}', \vec{0})$$

$$= \frac{1+a_2a_4}{a_1}r_5 - c_4p_4p_9 + c_5p_1p_9 + c_6(p_3p_4 - p_1p_6)$$
$$-\frac{a_4p_1+\frac{1+a_2a_4}{a_1}p_4}{a_1p_1+a_2p_4}(a_2r_5 - c_1p_4p_9 + c_3(p_3p_4 - p_1p_6))$$
$$= (\frac{1+a_2a_4}{a_1} - a_2\frac{a_4p_1+\frac{1+a_2a_4}{a_1}p_4}{a_1p_1+a_2p_4})r_5 + \text{rational}(\vec{P})$$
$$= \frac{p_1}{a_1p_1+a_2p_4}r_5 + \text{rational}(\vec{P}).$$

Since

$$\vec{r}'_6 = a_4r_3 + \frac{1+a_2a_4}{a_1}r_6 + c_6$$
$$= R_6(\vec{P}', \vec{R}', \vec{S})$$
$$= R_6(\vec{P}', \vec{R}', \vec{0}) - \frac{p'_4}{p'_9} \cdot 0 - \frac{1+p'_2p'_4}{p'_1p'_9} \cdot 0,$$

it follows

$$R_6(\vec{P}', \vec{R}', \vec{0})$$

$$= a_4r_3 + \frac{1+a_2a_4}{a_1}r_6 + c_6.$$

Thus we obtain all $R_k(\vec{P}', \vec{R}', \vec{0})$, $k = 1, 3, 4, 5, 6$, and hence we get the corresponding $\mathcal{A}' \in \mathrm{Ad}_G(ss_0)$ associated with $\tilde{\mathcal{A}} = \mathcal{A}\mathcal{A}_0$ such that $\mathcal{A}' \in C_S$.

Let's calculate both $\det\left[\frac{\partial p_i'}{\partial p_j}\right]_{6\times 6}$ and $\det\left[\frac{\partial R_i}{\partial r_j}\right]_{5\times 5}$.

$$
\begin{aligned}
& \det\left[\frac{\partial p_i'}{\partial p_j}\right]_{6\times 6} \\
= \ & \det\begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ -a_2\frac{1+p_2p_4}{p_1^2} & a_1 + a_2\frac{p_4}{p_1} & 0 & a_2\frac{p_2}{p_1} & 0 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 \\ a_4 & 0 & 0 & \frac{1+a_2a_4}{a_1} & 0 & 0 \\ 0 & 0 & a_4 & 0 & \frac{1+a_2a_4}{a_1} & a_6 \\ 0 & 0 & 0 & 0 & 0 & a_9 \end{bmatrix} \\
= \ & a_9(a_1 + a_2\frac{p_4}{p_1})\det\begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_4 & 0 & \frac{1+a_2a_4}{a_1} & 0 \\ 0 & a_4 & 0 & \frac{1+a_2a_4}{a_1} \end{bmatrix} \\
= \ & a_9(a_1 + a_2\frac{p_4}{p_1})\left\{a_1\det\begin{bmatrix} a_1 & 0 & a_2 \\ 0 & \frac{1+a_2a_4}{a_1} & 0 \\ a_4 & 0 & \frac{1+a_2a_4}{a_1} \end{bmatrix} + a_2\det\begin{bmatrix} 0 & a_1 & a_2 \\ a_4 & 0 & 0 \\ 0 & a_4 & \frac{1+a_2a_4}{a_1} \end{bmatrix}\right\} \\
= \ & a_9(a_1 + a_2\frac{p_4}{p_1})\left\{a_1(\frac{(1+a_2a_4)^2}{a_1} - \frac{a_2a_4(1+a_2a_4)}{a_1}) + a_2(a_2a_4^2 - a_4(1+a_2a_4))\right\} \\
= \ & a_9\frac{a_1p_1 + a_2p_4}{p_1}\left\{(1+a_2a_4) - a_2a_4\right\} \\
= \ & (\det A)\frac{p_1'}{p_1} \\
= \ & \frac{\det P'}{\det P}\frac{p_1'}{p_1}
\end{aligned}
$$

$$
\begin{aligned}
&\det\left[\frac{\partial R_i}{\partial r_j}\right]_{5\times 5} \\
&= \det\begin{bmatrix} a_1 & 0 & a_2 & * & 0 \\ 0 & a_1 & 0 & 0 & a_2 \\ a_4 & 0 & \frac{1+a_2a_4}{a_1} & * & 0 \\ 0 & 0 & 0 & \frac{p_1}{a_1p_1+a_2p_4} & 0 \\ 0 & a_4 & 0 & 0 & \frac{1+a_2a_4}{a_1} \end{bmatrix} \\
&= \frac{p_1}{a_1p_1+a_2p_4}\det\begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_4 & 0 & \frac{1+a_2a_4}{a_1} & 0 \\ 0 & a_4 & 0 & \frac{1+a_2a_4}{a_1} \end{bmatrix} \\
&= \frac{p_1}{a_1p_1+a_2p_4}\left\{a_1\det\begin{bmatrix} a_1 & 0 & a_2 \\ 0 & \frac{1+a_2a_4}{a_1} & 0 \\ a_4 & 0 & \frac{1+a_2a_4}{a_1} \end{bmatrix} + a_2\det\begin{bmatrix} 0 & a_1 & a_2 \\ a_4 & 0 & 0 \\ 0 & a_4 & \frac{1+a_2a_4}{a_1} \end{bmatrix}\right\} \\
&= \frac{p_1}{a_1p_1+a_2p_4} \\
&= \frac{p_1}{p_1'}
\end{aligned}
$$

Hence the determinant of the Jacobian of the map $\mathcal{A} \longmapsto \mathcal{A}'$ at each point $\mathcal{A}$ is

$$
\begin{aligned}
&\det\left[\frac{\partial(p_1',p_2',p_3',p_4',p_6',p_9',R_1,R_3,R_4,R_5,R_6)}{\partial(p_1,p_2,p_3,p_4,p_6,p_9,r_1,r_3,r_4,r_5,r_6)}\right] = \det\left[\frac{\partial p_i'}{\partial p_j}\right]_{6\times 6}\det\left[\frac{\partial R_i}{\partial r_j}\right]_{5\times 5} \\
&= \frac{(\det P')p_1'}{(\det P)p_1}\frac{p_1}{p_1'} = \frac{\det P'}{\det P} = \frac{p_9'}{p_9}
\end{aligned}
$$

Therefore $\frac{1}{p_9}dp_1dp_2dp_3dp_4dp_6dp_9dr_1dr_3dr_4dr_5dr_6$ is a right Haar measure on $\mathrm{Ad}_G\backslash\mathrm{Ad}_G\cdot\mathrm{Stab}^*(Z_3^*)$.

Lemma 5.3. Let $\mathfrak{g} = \mathcal{F}_{3,2}$, choose a basis $\mathcal{B}$ for $\mathfrak{g}$ like we did in Construction 1. Let C_A be the collection of all automorphisms $\mathcal{A}$ such that

$$[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} p_5p_9 - p_6p_8 & -(p_2p_9 - p_3p_8) & p_2p_6 - p_3p_5 & r_1 & r_4 & 0 \\ -(p_4p_9 - p_6p_7) & p_1p_9 - p_3p_7 & -(p_1p_6 - p_3p_4) & 0 & r_5 & 0 \\ p_4p_8 - p_5p_7 & -(p_1p_8 - p_2p_7) & p_1p_5 - p_2p_4 & r_3 & r_6 & r_9 \\ \hline 0 & 0 & 0 & p_1 & p_4 & p_7 \\ 0 & 0 & 0 & p_2 & p_5 & p_8 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

$\forall p_i, r_j \in \mathbf{R}$ with $p_9(p_1p_9 - p_3p_7)(\det P) \neq 0$, $i = 1, 2, \cdots, 9$, $j = 1, 3, 4, 5, 6, 9$. Claim that C_A is an almost global coordinate patch for $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})$.

Proof. For every $X \in \mathfrak{g}$, let $X = \sum_{i=1}^{3} z_i Z_i + \sum_{i=1}^{3} y_i Y_i$. Since $\mathfrak{g}$ is two-step, we showed already in Lemma 4.1. relative to the basis $\mathcal{B}$, $\mathrm{Ad}_{\exp(X)}$ has matrix

$$\left[\mathrm{Ad}_{\exp(X)}\right]_{\mathcal{B}} = \left[\begin{array}{c|c} I_{3\times 3} & S_{3\times 3} \\ \hline 0_{3\times 3} & I_{3\times 3} \end{array}\right] \text{ where } S_{3\times 3} = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}.$$

For each $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, let $[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{c|c} Q_{3\times 3} & R_{3\times 3} \\ \hline 0_{3\times 3} & P_{3\times 3} \end{array}\right]$ where $P_{3\times 3} = \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix}$,

$$Q_{3\times 3} = \begin{bmatrix} p_5p_9 - p_6p_8 & -(p_2p_9 - p_3p_8) & p_2p_6 - p_3p_5 \\ -(p_4p_9 - p_6p_7) & p_1p_9 - p_3p_7 & -(p_1p_6 - p_3p_4) \\ p_4p_8 - p_5p_7 & -(p_1p_8 - p_2p_7) & p_1p_5 - p_2p_4 \end{bmatrix}, \text{ and}$$

$$R_{3\times 3} = \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix}. \text{ Then}$$

$$[\mathrm{Ad}_{\exp(X)}\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{c|c} I & S \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} Q & R \\ \hline 0 & P \end{array}\right] = \left[\begin{array}{c|c} Q & R+SP \\ \hline 0 & P \end{array}\right],$$

and hence

$$\begin{aligned}
&\mathrm{Ad}_G\mathcal{A} \\
&= \{\mathrm{Ad}_{\exp(X)}\mathcal{A} \mid X = \textstyle\sum_{i=1}^{3} z_i Z_i + \sum_{i=1}^{3} y_i Y_i,\ \forall y_i, z_i \in \mathbf{R}\} \\
&= \left\{\mathrm{Ad}_{\exp(X)}\mathcal{A} \mid [\mathrm{Ad}_{\exp(X)}\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{c|c} Q & R+SP \\ \hline 0 & P \end{array}\right],\ \forall y_i \in \mathbf{R},\ i = 1,2,3\right\}
\end{aligned}$$

For the 3×3 matrix

$$\begin{aligned}
&R + SP \\
&= \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix} + \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix} \\
&= \begin{bmatrix} r_1 + p_3y_2 - p_2y_3 & r_4 + p_6y_2 - p_5y_3 & r_7 + p_9y_2 - p_8y_3 \\ r_2 - p_3y_1 + p_1y_3 & r_5 - p_6y_1 + p_4y_2 & r_8 - p_9y_1 + p_7y_3 \\ r_3 + p_2y_1 - p_1y_2 & r_6 + p_5y_1 - p_4y_2 & r_9 + p_8y_1 - p_7y_2 \end{bmatrix}
\end{aligned}$$

In order to re-coordinate the matrix $R + SP$, we need to put restrictions on some entries of P. We choose the condition for P is

$$p_9 \neq 0 \text{ and } \delta := \det \begin{bmatrix} p_1 & p_7 \\ p_3 & p_9 \end{bmatrix} \neq 0$$

Then let

$$\begin{aligned}
s_1 &= r_2 - p_3y_1 + p_1y_3 \\
s_2 &= r_8 - p_9y_1 + p_7y_3 \\
s_3 &= r_7 + p_9y_2 - p_8y_3
\end{aligned}$$

Under the restrictions $p_9 \neq 0$ and $\delta = p_1p_9 - p_3p_7 \neq 0$, we get

$$\begin{aligned}
y_1 &= (\frac{-p_7}{\delta}r_2 + \frac{p_1}{\delta}r_8) + \frac{p_7}{\delta}s_1 - \frac{p_1}{\delta}s_2 \\
y_2 &= (\frac{-p_8}{\delta}r_2 - \frac{1}{p_9}r_7 + \frac{p_3p_8}{p_9\delta}r_8) + \frac{p_8}{\delta}s_1 + \frac{-p_3p_8}{p_9\delta}s_2 + \frac{1}{p_9}s_3 \\
y_3 &= (\frac{-p_9}{\delta}r_2 + \frac{p_3}{\delta}r_8) + \frac{p_9}{\delta}s_1 - \frac{p_3}{\delta}s_2
\end{aligned}$$

Thus, we replace y_1, y_2, and y_3 by the above corresponding identity in each entry of $R + SP$, and denote

$\vec{P} := \vec{P}(p_1, p_2, \cdots, p_9)$ corresponding to matrix $P_{3\times3}$,

$\vec{R} := \vec{R}(r_1, r_2, \cdots, r_9)$ corresponding to matrix $R_{3\times3}$,

$\vec{S} := \vec{S}(s_1, s_2, s_3)$ corresponding to triple (s_1, s_2, s_3).

Then we can denote matrix

$$R + SP = \begin{bmatrix} R_1(\vec{P}, \vec{R}, \vec{S}) & R_4(\vec{P}, \vec{R}, \vec{S}) & s_3 \\ s_1 & R_5(\vec{P}, \vec{R}, \vec{S}) & s_2 \\ R_3(\vec{P}, \vec{R}, \vec{S}) & R_6(\vec{P}, \vec{R}, \vec{S}) & R_9(\vec{P}, \vec{R}, \vec{S}) \end{bmatrix}$$

, and hence

$$\begin{aligned}
&\mathrm{Ad}_G\mathcal{A} \\
&\doteq \{\mathrm{Ad}_g\mathcal{A} | [\mathrm{Ad}_g\mathcal{A}]_{\mathcal{B}} = \begin{bmatrix} Q & \begin{pmatrix} R_1(\vec{P}, \vec{R}, \vec{S}) & R_4(\vec{P}, \vec{R}, \vec{S}) & s_3 \\ s_1 & R_5(\vec{P}, \vec{R}, \vec{S}) & s_2 \\ R_3(\vec{P}, \vec{R}, \vec{S}) & R_6(\vec{P}, \vec{R}, \vec{S}) & R_9(\vec{P}, \vec{R}, \vec{S}) \end{pmatrix} \\ 0 & P \end{bmatrix}, \vec{S} \in \\
&\mathbf{R}^3\}
\end{aligned}$$

Let's figure out the explicit formula for each $R_k(\vec{P}, \vec{R}, \vec{S})$, $k = 1, 3, 4, 5, 6, 9$.

$$R_1(\vec{P},\vec{R},\vec{S})$$

$$= r_1 + p_3y_2 - p_2y_3$$

$$= r_1 + p_3((\tfrac{-p_8}{\delta}r_2 - \tfrac{1}{p_9}r_7 + \tfrac{p_3p_8}{p_9\delta}r_8) + \tfrac{p_8}{\delta}s_1 + \tfrac{-p_3p_8}{p_9\delta}s_2 + \tfrac{1}{p_9}s_3)$$

$$-p_2((\tfrac{-p_9}{\delta}r_2 + \tfrac{p_3}{\delta}r_8) + \tfrac{p_9}{\delta}s_1 - \tfrac{p_3}{\delta}s_2)$$

$$= (r_1 + \tfrac{p_2p_9-p_3p_8}{\delta}r_2 - \tfrac{p_3}{p_9}r_7 - \tfrac{p_3(p_2p_9-p_3p_8)}{p_9\delta}r_8)$$

$$-\tfrac{p_2p_9-p_3p_8}{\delta}s_1 + \tfrac{p_3}{p_9}s_3 + \tfrac{p_3(p_2p_9-p_3p_8)}{p_9\delta}s_2,$$

$$R_3(\vec{P},\vec{R},\vec{S})$$

$$= r_3 + p_2y_1 - p_1y_2$$

$$= r_3 + p_2((\tfrac{-p_7}{\delta}r_2 + \tfrac{p_1}{\delta}r_8) + \tfrac{p_7}{\delta}s_1 - \tfrac{p_1}{\delta}s_2)$$

$$-p_1((\tfrac{-p_8}{\delta}r_2 - \tfrac{1}{p_9}r_7 + \tfrac{p_3p_8}{p_9\delta}r_8) + \tfrac{p_8}{\delta}s_1 + \tfrac{-p_3p_8}{p_9\delta}s_2 + \tfrac{1}{p_9}s_3)$$

$$= (\tfrac{p_1p_8-p_2p_7}{\delta}r_2 + r_3 + \tfrac{p_1}{p_9}r_7 + \tfrac{p_1(p_2p_9-p_3p_8)}{p_9\delta}r_8)$$

$$-\tfrac{p_1p_8-p_2p_7}{\delta}s_1 - \tfrac{p_1}{p_9}s_3 - \tfrac{p_1(p_2p_9-p_3p_8)}{p_9\delta}s_2,$$

$$R_4(\vec{P},\vec{R},\vec{S})$$

$$= r_4 + p_6y_2 - p_5y_3$$

$$= r_4 + p_6((\tfrac{-p_8}{\delta}r_2 - \tfrac{1}{p_9}r_7 + \tfrac{p_3p_8}{p_9\delta}r_8) + \tfrac{p_8}{\delta}s_1 + \tfrac{-p_3p_8}{p_9\delta}s_2 + \tfrac{1}{p_9}s_3)$$

$$-p_5((\tfrac{-p_9}{\delta}r_2 + \tfrac{p_3}{\delta}r_8) + \tfrac{p_9}{\delta}s_1 - \tfrac{p_3}{\delta}s_2)$$

$$= (\tfrac{p_5p_9-p_6p_8}{\delta}r_2 + r_4 - \tfrac{p_6}{p_9}r_7 - \tfrac{p_3(p_5p_9-p_6p_8)}{p_9\delta}r_8)$$

$$-\tfrac{p_5p_9-p_6p_8}{\delta}s_1 + \tfrac{p_6}{p_9}s_3 + \tfrac{p_3(p_5p_9-p_6p_8)}{p_9\delta}s_2,$$

$$R_5(\vec{P},\vec{R},\vec{S})$$

$$= r_5 - p_6y_1 + p_4y_3$$

$$= r_5 - p_6((\tfrac{-p_7}{\delta}r_2 + \tfrac{p_1}{\delta}r_8) + \tfrac{p_7}{\delta}s_1 - \tfrac{p_1}{\delta}s_2)$$

$$+p_4((\tfrac{-p_9}{\delta}r_2 + \tfrac{p_3}{\delta}r_8) + \tfrac{p_9}{\delta}s_1 - \tfrac{p_3}{\delta}s_2)$$

$$= (-\tfrac{p_4p_9-p_6p_7}{\delta}r_2 + r_5 - \tfrac{p_1p_6-p_3p_4}{\delta}r_8)$$

$$+\tfrac{p_4p_9-p_6p_7}{\delta}s_1 + \tfrac{p_1p_6-p_3p_4}{\delta}s_2,$$

$R_6(\vec{P},\vec{R},\vec{S})$

$= r_6 + p_5y_1 - p_4y_2$

$= r_6 + p_5((\frac{-p_7}{\delta}r_2 + \frac{p_1}{\delta}r_8) + \frac{p_7}{\delta}s_1 - \frac{p_1}{\delta}s_2)$

$-p_4((\frac{-p_8}{\delta}r_2 - \frac{1}{p_9}r_7 + \frac{p_3p_8}{p_9\delta}r_8) + \frac{p_8}{\delta}s_1 + \frac{-p_3p_8}{p_9\delta}s_2 + \frac{1}{p_9}s_3)$

$= (\frac{p_4p_8 - p_5p_7}{\delta}r_2 + r_6 + \frac{p_4}{p_9}r_7 + \frac{p_1p_5p_9 - p_3p_4p_8}{p_9\delta}r_8)$

$-\frac{p_4p_8 - p_5p_7}{\delta}s_1 - \frac{p_4}{p_9}s_3 - \frac{p_1p_5p_9 - p_3p_4p_8}{p_9\delta}s_2,$

$R_9(\vec{P},\vec{R},\vec{S})$

$= r_9 + p_8y_1 - p_7y_2$

$= r_9 + p_8((\frac{-p_7}{\delta}r_2 + \frac{p_1}{\delta}r_8) + \frac{p_7}{\delta}s_1 - \frac{p_1}{\delta}s_2)$

$-p_7((\frac{-p_8}{\delta}r_2 - \frac{1}{p_9}r_7 + \frac{p_3p_8}{p_9\delta}r_8) + \frac{p_8}{\delta}s_1 + \frac{-p_3p_8}{p_9\delta}s_2 + \frac{1}{p_9}s_3)$

$= (\frac{p_7}{p_9}r_7 + \frac{p_8}{p_9}r_8 + r_9) - \frac{p_8}{p_9}s_2 - \frac{p_7}{p_9}s_3.$

Since $\vec{S}$ varying all $\mathbf{R}^3$ makes a coset $\mathrm{Ad}_G\mathcal{A}$, pick $\vec{S} = \vec{0}$, then we have unique solution (y_1, y_2, y_3) where

$$\begin{aligned} y_1 &= \frac{-p_7}{p_1p_9 - p_3p_7}r_2 + \frac{p_1}{p_1p_9 - p_3p_7}r_8 \\ y_2 &= \frac{-p_8}{p_1p_9 - p_3p_7}r_2 - \frac{1}{p_9}r_7 + \frac{p_3p_8}{p_9(p_1p_9 - p_3p_7)}r_8 \\ y_3 &= \frac{-p_9}{p_1p_9 - p_3p_7}r_2 + \frac{p_3}{p_1p_9 - p_3p_7}r_8 \end{aligned}$$

to make a unique automorphism $\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_iY_i)}\mathcal{A} \in \mathrm{Ad}_G\mathcal{A}$ for each $\mathcal{A} \in \mathrm{Aut}(\mathrm{g})$ satisfying the restriction $p_9(p_1p_9 - p_3p_7) \neq 0$ with

$$\left[\mathrm{Ad}_{\exp(\sum_{i=1}^3 y_iY_i)}\mathcal{A}\right]_B = \begin{bmatrix} Q_{3\times3} & \begin{pmatrix} R_1(\vec{P},\vec{R}) & R_4(\vec{P},\vec{R}) & 0 \\ 0 & R_5(\vec{P},\vec{R}) & 0 \\ R_3(\vec{P},\vec{R}) & R_6(\vec{P},\vec{R}) & R_9(\vec{P},\vec{R}) \end{pmatrix} \\ 0_{3\times3} & P_{3\times3} \end{bmatrix}$$

where each $R_k(\vec{P},\vec{R}) = R_k(\vec{P},\vec{R},\vec{0})$, $k = 1,3,4,5,6,9$.

$R_1(\vec{P},\vec{R}) = r_1 + \frac{p_2p_9-p_3p_8}{p_1p_9-p_3p_7}r_2 - \frac{p_3}{p_9}r_7 - \frac{p_3(p_2p_9-p_3p_8)}{p_9(p_1p_9-p_3p_7)}r_8$

$R_3(\vec{P},\vec{R}) = \frac{p_1p_8-p_2p_7}{p_1p_9-p_3p_7}r_2 + r_3 + \frac{p_1}{p_9}r_7 + \frac{p_1(p_2p_9-p_3p_8)}{p_9(p_1p_9-p_3p_7)}r_8$

$R_4(\vec{P},\vec{R}) = \frac{p_5p_9-p_6p_8}{p_1p_9-p_3p_7}r_2 + r_4 - \frac{p_6}{p_9}r_7 - \frac{p_3(p_5p_9-p_6p_8)}{p_9(p_1p_9-p_3p_7)}r_8$

$R_5(\vec{P},\vec{R}) = -\frac{p_4p_9-p_6p_7}{p_1p_9-p_3p_7}r_2 + r_5 - \frac{p_1p_6-p_3p_4}{p_1p_9-p_3p_7}r_8$

$R_6(\vec{P},\vec{R}) = \frac{p_4p_8-p_5p_7}{p_1p_9-p_3p_7}r_2 + r_6 + \frac{p_4}{p_9}r_7 + \frac{p_1p_5p_9-p_3p_4p_8}{p_9(p_1p_9-p_3p_7)}r_8$

$R_9(\vec{P},\vec{R}) = \frac{p_7}{p_9}r_7 + \frac{p_8}{p_9}r_8 + r_9$

Thus, as $\vec{P}(p_1,p_2,\cdots,p_9)$ and $\vec{R}(r_1,r_2,\cdots,r_9)$ vary over all $\mathbf{R}^9$ satisfying $p_9(p_1p_9-p_3p_7)(\det P) \neq 0$, each $R_k(\vec{P},\vec{R})$ varies over all $\mathbf{R}$, $k = 1,3,4,5,6,9$.

Now let C_A be the collection of all automorphisms $\mathcal{A}$ of $\mathfrak{g}$ such that relative to the basis $\mathcal{B}$, $\mathcal{A}$ has matrix

$$[\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{ccc|ccc} p_5p_9-p_6p_8 & -(p_2p_9-p_3p_8) & p_2p_6-p_3p_5 & R_1 & R_4 & 0 \\ -(p_4p_9-p_6p_7) & p_1p_9-p_3p_7 & -(p_1p_6-p_3p_4) & 0 & R_5 & 0 \\ p_4p_8-p_5p_7 & -(p_1p_8-p_2p_7) & p_1p_5-p_2p_4 & R_3 & R_6 & R_9 \\ \hline 0 & 0 & 0 & p_1 & p_4 & p_7 \\ 0 & 0 & 0 & p_2 & p_5 & p_8 \\ 0 & 0 & 0 & p_3 & p_6 & p_9 \end{array}\right]$$

$\forall p_i, R_k \in \mathbf{R}$ satisfying $p_9(p_1p_9-p_3p_7) \neq 0$, $i = 1,2,\cdots,9$, $k = 1,3,4,5,6,9$. Then we verify that different elements of C_S lie on different cosets. If this is done, that means C_S is an almost global coordinate patch for $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})$.

For every distinct $c, c' \in C_A$, we know $c \in \mathrm{Ad}_G\mathcal{A}$ and $c' \in \mathrm{Ad}_G\mathcal{A}'$ for some $\mathcal{A}, \mathcal{A}' \in \mathrm{Aut}(\mathfrak{g})$. Let

$$[c]_{\mathcal{B}} = \left[\begin{array}{cc} Q_{3\times 3} & \begin{pmatrix} R_1 & R_4 & 0 \\ 0 & R_5 & 0 \\ R_3 & R_6 & R_9 \end{pmatrix} \\ 0_{3\times 3} & P_{3\times 3} \end{array}\right] \text{ and } [c']_{\mathcal{B}} = \left[\begin{array}{cc} Q'_{3\times 3} & \begin{pmatrix} R'_1 & R'_4 & 0 \\ 0 & R'_5 & 0 \\ R'_3 & R'_6 & R'_9 \end{pmatrix} \\ 0_{3\times 3} & P'_{3\times 3} \end{array}\right]$$

like the above matrix in C_A.

Then $\mathrm{Ad}_G\mathcal{A}$ is the set of all $\mathrm{Ad}_g\mathcal{A}$ such that

$$[\mathrm{Ad}_g\mathcal{A}]_{\mathcal{B}} = \left[\begin{array}{cc} Q & \begin{pmatrix} R_1(\vec{P},\vec{R},\vec{S}) & R_4(\vec{P},\vec{R},\vec{S}) & s_3 \\ s_1 & R_5(\vec{P},\vec{R},\vec{S}) & s_2 \\ R_3(\vec{P},\vec{R},\vec{S}) & R_6(\vec{P},\vec{R},\vec{S}) & R_9(\vec{P},\vec{R},\vec{S}) \end{pmatrix} \\ 0 & P \end{array}\right], \forall \vec{S} \in \mathbf{R}^3.$$

And it's similar for $\mathrm{Ad}_G\mathcal{A}'$. Since $c \neq c'$, it requires at least one $p_i \neq p'_i$ or $R_j \neq R'_j$. If $p_i \neq p'_i$ for some $i \in \{1, 2, \cdots, 9\}$, then $\mathrm{Ad}_G\mathcal{A} \neq \mathrm{Ad}_G\mathcal{A}'$. Suppose $p_i = p'_i$ for all i, i.e., $\vec{P} = \vec{P}'$. If $R_j \neq R'_j$ for some $j \in \{1, 3, 4, 5, 6, 9\}$. For example, let $R_1 \neq R'_1$. This means $R_1(\vec{P},\vec{R},\vec{0}) \neq R'_1(\vec{P}',\vec{R}',\vec{0})$, then $R_1(\vec{P},\vec{R},\vec{0}) \neq R'_1(\vec{P},\vec{R}',\vec{0})$. By the formula of $R_1(\vec{P},\vec{R},\vec{S})$, we get

$$\begin{aligned} R_1(\vec{P},\vec{R},\vec{S}) &= R_1(\vec{P},\vec{R},\vec{0}) - \tfrac{p_2p_9 - p_3p_8}{p_1p_9 - p_3p_7}s_1 + \tfrac{p_3}{p_9}s_3 + \tfrac{p_3(p_2p_9 - p_3p_8)}{p_9(p_1p_9 - p_3p_7)}s_2 \\ &\neq R'_1(\vec{P},\vec{R}',\vec{0}) - \frac{p_2p_9 - p_3p_8}{p_1p_9 - p_3p_7}s_1 + \frac{p_3}{p_9}s_3 + \frac{p_3(p_2p_9 - p_3p_8)}{p_9(p_1p_9 - p_3p_7)}s_2 \\ &= R'_1(\vec{P},\vec{R}',\vec{S}) \\ &= R'_1(\vec{P}',\vec{R}',\vec{S}) \end{aligned}$$

So $R_1(\vec{P},\vec{R},\vec{S}) \neq R'_1(\vec{P}',\vec{R}',\vec{S})$, and hence $\mathrm{Ad}_G\mathcal{A} \neq \mathrm{Ad}_G\mathcal{A}'$. Similarly for other cases $R_j \neq R'_j$, $j = 3, 4, 5, 6, 9$, we obtain the same result. Therefore different elements of C_A lie on different cosets. One concludes C_A is an almost global coordinate patch for $\mathrm{Ad}_G\backslash\mathrm{Aut}(\mathfrak{g})$.

Lemma 5.4. The assumption is the same as Lemma 5.3. With respect to the coordinate patch C_A, we claim that

$$\frac{|p_9(p_1p_9-p_3p_7)|}{(\det P)^8}dp_1dp_2\cdots dp_9dr_1dr_3dr_4dr_5dr_6dr_9$$

is a left Haar measure and

$$\frac{|p_9(p_1p_9-p_3p_7)|}{(\det P)^8}dp_1dp_2\cdots dp_9dr_1dr_3dr_4dr_5dr_6dr_9$$

is a right Haar measure on $\mathrm{Ad}_G\backslash\mathrm{Aut}(\mathfrak{g})$.

Proof. Given $\mathcal{A}_0 \in C_A$, for each $\mathcal{A} \in C_A$, let $\mathcal{A}' = \mathcal{A}_0\mathcal{A}$. But $\mathcal{A}'$ may not be in the patch C_A, we would like to find the corresponding $\tilde{\mathcal{A}} \in \mathrm{Ad}_G\mathcal{A}'$ such that $\tilde{\mathcal{A}}$ is in C_A. Then we can calculate the determinant of the Jacobian of the map $\mathcal{A} \longmapsto \tilde{\mathcal{A}}$ at point $\mathcal{A}$ in order to figure out a left Haar measure on $\mathrm{Ad}_G\backslash\mathrm{Aut}(\mathfrak{g})$.

Since $\mathcal{A}_0, \mathcal{A} \in C_A$, let

$$[\mathcal{A}_0]_{\mathcal{B}} = \begin{bmatrix} B_{3\times 3} & \begin{pmatrix} c_1 & c_4 & 0 \\ 0 & c_5 & 0 \\ c_3 & c_6 & c_9 \end{pmatrix} \\ 0_{3\times 3} & (a_{ij})_{3\times 3} \end{bmatrix} =: \left[\begin{array}{c|c} B_{3\times 3} & C_{3\times 3} \\ \hline 0_{3\times 3} & A_{3\times 3} \end{array}\right] \text{ and}$$

$$[\mathcal{A}]_{\mathcal{B}} = \begin{bmatrix} Q_{3\times 3} & \begin{pmatrix} r_1 & r_4 & 0 \\ 0 & r_5 & 0 \\ r_3 & r_6 & r_9 \end{pmatrix} \\ 0_{3\times 3} & (p_{ij})_{3\times 3} \end{bmatrix} =: \left[\begin{array}{c|c} Q_{3\times 3} & R_{3\times 3} \\ \hline 0_{3\times 3} & P_{3\times 3} \end{array}\right] \text{ where matrices B and Q}$$

are determined by A and P respectively. Then

$$[\mathcal{A}']_{\mathcal{B}} = \left[\begin{array}{c|c} B & C \\ \hline 0 & A \end{array}\right]\left[\begin{array}{c|c} Q & R \\ \hline 0 & P \end{array}\right] = \left[\begin{array}{c|c} BQ & BR+CP \\ \hline 0 & AP \end{array}\right] =: \left[\begin{array}{c|c} Q'_{3\times 3} & R'_{3\times 3} \\ \hline 0_{3\times 3} & P'_{3\times 3} \end{array}\right]$$

and hence from previous lemma we know the unique $\tilde{\mathcal{A}} \in \mathrm{Ad}_G\mathcal{A}'$ such that $\tilde{\mathcal{A}}$ is in C_A which its matrix is

$$
\left[\tilde{A}\right]_{\mathcal{B}} = \begin{bmatrix} Q'_{3\times3} & \begin{pmatrix} R_1(\vec{P}',\vec{R}') & R_4(\vec{P}',\vec{R}') & 0 \\ 0 & R_5(\vec{P}',\vec{R}') & 0 \\ R_3(\vec{P}',\vec{R}') & R_6(\vec{P}',\vec{R}') & R_9(\vec{P}',\vec{R}') \end{pmatrix} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}
$$

So we need to figure out matrices P' and R', then we will know each $R_i(\vec{P}',\vec{R}')$, $i = 1,3,4,5,6,9$. For the matrices

$$
\begin{aligned}
& P' \\
= \; & AP \\
= \; & \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix} \\
= \; & \begin{bmatrix} a_1p_1 + a_4p_2 + a_7p_3 & a_1p_4 + a_4p_5 + a_7p_6 & a_1p_7 + a_4p_8 + a_7p_9 \\ a_2p_1 + a_5p_2 + a_8p_3 & a_2p_4 + a_5p_5 + a_8p_6 & a_2p_7 + a_5p_8 + a_8p_9 \\ a_3p_1 + a_6p_2 + a_9p_3 & a_3p_4 + a_6p_5 + a_9p_6 & a_3p_7 + a_6p_8 + a_9p_9 \end{bmatrix} \\
= : & \begin{bmatrix} p'_1 & p'_4 & p'_7 \\ p'_2 & p'_5 & p'_8 \\ p'_3 & p'_6 & p'_9 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& R' \\
= \; & BR + CP \\
= \; & \begin{bmatrix} b_1 & b_4 & b_7 \\ b_2 & b_5 & b_8 \\ b_3 & b_6 & b_9 \end{bmatrix} \begin{bmatrix} r_1 & r_4 & 0 \\ 0 & r_5 & 0 \\ r_3 & r_6 & r_9 \end{bmatrix} + \begin{bmatrix} c_1 & c_4 & 0 \\ 0 & c_5 & 0 \\ c_3 & c_6 & c_9 \end{bmatrix} \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix} \\
= : & \begin{bmatrix} r'_1 & r'_4 & r'_7 \\ r'_2 & r'_5 & r'_8 \\ r'_3 & r'_6 & r'_9 \end{bmatrix}
\end{aligned}
$$

where

$r'_1 = b_1r_1 + b_7r_3 + c_1p_1 + c_4p_2,$

$r'_2 = b_2r_1 + b_8r_3 + c_5p_2,$

$r'_3 = b_3r_1 + b_9r_3 + c_3p_1 + c_6p_2 + c_9p_3,$

$r'_4 = b_1r_4 + b_4r_5 + b_7r_6 + c_1p_4 + c_4p_5,$

$r'_5 = b_2r_4 + b_5r_5 + b_8r_6 + c_5p_5,$

$r'_6 = b_3r_4 + b_6r_5 + b_9r_6 + c_3p_4 + c_6p_5 + c_9p_6,$

$r'_7 = b_7r_9 + c_1p_7 + c_4p_8,$

$r'_8 = b_8r_9 + c_5p_8,$

$r'_9 = b_9r_9 + c_3p_7 + c_6p_8 + c_9p_9.$

Let $\delta' := p'_1p'_9 - p'_3p'_7$. By formulas in previous lemma we get each $R_i(\vec{P}', \vec{R}')$:

$$\begin{aligned}
&R_1(\vec{P}', \vec{R}') \\
&= r'_1 + \tfrac{p'_2p'_9 - p'_3p'_8}{\delta'} r'_2 - \tfrac{p'_3}{p'_9} r'_7 - \tfrac{p'_3(p'_2p'_9 - p'_3p'_8)}{p'_9\delta'} r'_8 \\
&= (b_1r_1 + b_7r_3 + c_1p_1 + c_4p_2) + \tfrac{p'_2p'_9 - p'_3p'_8}{\delta'}(b_2r_1 + b_8r_3 + c_5p_2) \\
&\quad - \tfrac{p'_3}{p'_9}(b_7r_9 + c_1p_7 + c_4p_8) - \tfrac{p'_3(p'_2p'_9 - p'_3p'_8)}{p'_9\delta'}(b_8r_9 + c_5p_8) \\
&= (b_1 + b_2\tfrac{p'_2p'_9 - p'_3p'_8}{\delta'})r_1 + (b_7 + b_8\tfrac{p'_2p'_9 - p'_3p'_8}{\delta'})r_3 \\
&\quad - (b_7\tfrac{p'_3}{p'_9} + b_8\tfrac{p'_3(p'_2p'_9 - p'_3p'_8)}{p'_9\delta'})r_9 + \text{rational}(\vec{P}),
\end{aligned}$$

$$\begin{aligned}
&R_3(\vec{P}', \vec{R}') \\
&= \tfrac{p'_1p'_8 - p'_2p'_7}{\delta'} r'_2 + r'_3 + \tfrac{p'_1}{p'_9} r'_7 + \tfrac{p'_1(p'_2p'_9 - p'_3p'_8)}{p'_9\delta'} r'_8 \\
&= \tfrac{p'_1p'_8 - p'_2p'_7}{\delta'}(b_2r_1 + b_8r_3 + c_5p_2) + (b_3r_1 + b_9r_3 + c_3p_1 + c_6p_2 + c_9p_3) \\
&\quad + \tfrac{p'_1}{p'_9}(b_7r_9 + c_1p_7 + c_4p_8) + \tfrac{p'_1(p'_2p'_9 - p'_3p'_8)}{p'_9\delta'}(b_8r_9 + c_5p_8) \\
&= (b_2\tfrac{p'_1p'_8 - p'_2p'_7}{\delta'} + b_3)r_1 + (b_8\tfrac{p'_1p'_8 - p'_2p'_7}{\delta'} + b_9)r_3 \\
&\quad + (b_7\tfrac{p'_1}{p'_9} + b_8\tfrac{p'_1(p'_2p'_9 - p'_3p'_8)}{p'_9\delta'})r_9 + \text{rational}(\vec{P}),
\end{aligned}$$

$$
\begin{aligned}
&R_4(\vec{P}', \vec{R}') \\
&= \tfrac{p'_5p'_9-p'_6p'_8}{\delta'}r'_2 + r'_4 - \tfrac{p'_8}{p'_9}r'_7 - \tfrac{p'_3(p'_5p'_9-p'_6p'_8)}{p'_9\delta'}r'_8 \\
&= \tfrac{p'_5p'_9-p'_6p'_8}{\delta'}(b_2r_1 + b_8r_3 + c_5p_2) + (b_1r_4 + b_4r_5 + b_7r_6 + c_1p_4 + c_4p_5) \\
&\quad -\tfrac{p'_8}{p'_9}(b_7r_9 + c_1p_7 + c_4p_8) - \tfrac{p'_3(p'_5p'_9-p'_6p'_8)}{p'_9\delta'}(b_8r_9 + c_5p_8) \\
&= (b_2\tfrac{p'_5p'_9-p'_6p'_8}{\delta'})r_1 + (b_8\tfrac{p'_5p'_9-p'_6p'_8}{\delta'})r_3 + b_1r_4 + b_4r_5 + b_7r_6 \\
&\quad -(b_7\tfrac{p'_8}{p'_9} + b_8\tfrac{p'_3(p'_5p'_9-p'_6p'_8)}{p'_9\delta'})r_9 + \text{rational}(\vec{P}),
\end{aligned}
$$

$$
\begin{aligned}
&R_5(\vec{P}', \vec{R}') \\
&= -\tfrac{p'_4p'_9-p'_6p'_7}{\delta'}r'_2 + r'_5 - \tfrac{p'_1p'_6-p'_3p'_4}{\delta'}r'_8 \\
&= -\tfrac{p'_4p'_9-p'_6p'_7}{\delta'}(b_2r_1 + b_8r_3 + c_5p_2) + (b_2r_4 + b_5r_5 + b_8r_6 + c_5p_5) \\
&\quad -\tfrac{p'_1p'_6-p'_3p'_4}{\delta'}(b_8r_9 + c_5p_8) \\
&= (-b_2\tfrac{p'_4p'_9-p'_6p'_7}{\delta'})r_1 + (-b_8\tfrac{p'_4p'_9-p'_6p'_7}{\delta'})r_3 + b_2r_4 + b_5r_5 + b_8r_6 \\
&\quad +(-b_8\tfrac{p'_1p'_6-p'_3p'_4}{\delta'})r_9 + \text{rational}(\vec{P}),
\end{aligned}
$$

$$
\begin{aligned}
&R_6(\vec{P}', \vec{R}') \\
&= \tfrac{p'_4p'_8-p'_5p'_7}{\delta'}r'_2 + r'_6 + \tfrac{p'_4}{p'_9}r'_7 + \tfrac{p'_1p'_5p'_9-p'_3p'_4p'_8}{p'_9\delta'}r'_8 \\
&= \tfrac{p'_4p'_8-p'_5p'_7}{\delta'}r'_2 + r'_6 + \tfrac{p'_4}{p'_9}r'_7 + \tfrac{p'_1p'_5p'_9-p'_3p'_4p'_8}{p'_9\delta'}r'_8 + b_9r_6 + c_3p_4 + c_6p_5 + c_9p_6) \\
&\quad +\tfrac{p'_4}{p'_9}(b_7r_9 + c_1p_7 + c_4p_8) + \tfrac{p'_1p'_5p'_9-p'_3p'_4p'_8}{p'_9\delta'}(b_8r_9 + c_5p_8) \\
&= (b_2\tfrac{p'_4p'_8-p'_5p'_7}{\delta'})r_1 + (b_8\tfrac{p'_4p'_8-p'_5p'_7}{\delta'})r_3 + b_3r_4 + b_6r_5 + b_9r_6 \\
&\quad +(b_7\tfrac{p'_4}{p'_9} + b_8\tfrac{p'_1p'_5p'_9-p'_3p'_4p'_8}{p'_9\delta'})r_9 + \text{rational}(\vec{P}),
\end{aligned}
$$

$$
\begin{aligned}
&R_9(\vec{P}', \vec{R}') \\
&= \tfrac{p'_7}{p'_9}r'_7 + \tfrac{p'_8}{p'_9}r'_8 + r'_9 \\
&= \tfrac{p'_7}{p'_9}(b_7r_9 + c_1p_7 + c_4p_8) + \tfrac{p'_8}{p'_9}(b_8r_9 + c_5p_8) + (b_9r_9 + c_3p_7 + c_6p_8 + c_9p_9) \\
&= (b_7\tfrac{p'_7}{p'_9} + b_8\tfrac{p'_8}{p'_9} + b_9)r_9 + \text{rational}(\vec{P}).
\end{aligned}
$$

Thus we obtain all $R_i(\vec{P}', \vec{R}')$, and hence we get unique $\tilde{\mathcal{A}} \in \mathrm{Ad}_G\mathcal{A}'$ such that $\tilde{\mathcal{A}}$ is in the patch C_A. Now let's calculate both $\det \left[\frac{\partial p'_i}{\partial p_j}\right]_{9\times 9}$ and $\det \left[\frac{\partial R_i}{\partial r_j}\right]_{6\times 6}$.

$$\det \left[\frac{\partial p'_i}{\partial p_j}\right]_{9\times 9}$$

$$= \det \begin{bmatrix} a_1 & a_4 & a_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_5 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_6 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_4 & a_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & a_5 & a_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & a_6 & a_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_4 & a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & a_5 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & a_6 & a_9 \end{bmatrix}$$

$$= (\det A_{3\times 3})^3$$

$$= \frac{(\det P')^3}{(\det P)^3}$$

$$\det \left[\frac{\partial R_i}{\partial r_j}\right]_{6\times 6}$$

$$= \det \begin{bmatrix} b_1 + b_2\frac{p'_2p'_9-p'_3p'_8}{\delta'} & b_7 + b_8\frac{p'_2p'_9-p'_3p'_8}{\delta'} & 0 & 0 & 0 & * \\ b_2\frac{p'_1p'_8-p'_2p'_7}{\delta'} + b_3 & b_8\frac{p'_1p'_8-p'_2p'_7}{\delta'} + b_9 & 0 & 0 & 0 & * \\ b_2\frac{p'_5p'_9-p'_6p'_8}{\delta'} & b_8\frac{p'_5p'_9-p'_6p'_8}{\delta'} & b_1 & b_4 & b_7 & * \\ -b_2\frac{p'_4p'_9-p'_6p'_7}{\delta'} & -b_8\frac{p'_4p'_9-p'_6p'_7}{\delta'} & b_2 & b_5 & b_8 & * \\ b_2\frac{p'_4p'_8-p'_5p'_7}{\delta'} & b_8\frac{p'_4p'_8-p'_5p'_7}{\delta'} & b_3 & b_6 & b_9 & * \\ 0 & 0 & 0 & 0 & 0 & b_7\frac{p'_7}{p'_9} + b_8\frac{p'_8}{p'_9} + b_9 \end{bmatrix}$$

$$= (b_7\frac{p'_7}{p'_9} + b_8\frac{p'_8}{p'_9} + b_9)(\det B_{3\times 3}) \det \begin{bmatrix} b_1 + b_2\frac{p'_2p'_9-p'_3p'_8}{\delta'} & b_7 + b_8\frac{p'_2p'_9-p'_3p'_8}{\delta'} \\ b_3 + b_2\frac{p'_1p'_8-p'_2p'_7}{\delta'} & b_9 + b_8\frac{p'_1p'_8-p'_2p'_7}{\delta'} \end{bmatrix}$$

$$= \frac{1}{p'_9}(b_7p'_7 + b_8p'_8 + b_9p'_9)(\det B_{3\times 3})$$

$$\cdot\left\{(b_1 + b_2\frac{p'_2p'_9-p'_3p'_8}{\delta'})(b_9 + b_8\frac{p'_1p'_8-p'_2p'_7}{\delta'}) - (b_3 + b_2\frac{p'_1p'_8-p'_2p'_7}{\delta'})(b_9 + b_8\frac{p'_1p'_8-p'_2p'_7}{\delta'})\right\}$$

Since

$$
\begin{aligned}
& b_7p'_7 + b_8p'_8 + b_9p'_9 \\
&= \begin{vmatrix} a_2 & a_5 \\ a_3 & a_6 \end{vmatrix} (a_1p_7 + a_4p_8 + a_7p_9) - \begin{vmatrix} a_1 & a_4 \\ a_3 & a_6 \end{vmatrix} (a_2p_7 + a_5p_8 + a_8p_9) \\
&\quad + \begin{vmatrix} a_1 & a_4 \\ a_2 & a_5 \end{vmatrix} (a_3p_7 + a_6p_8 + a_9p_9) \\
&= (a_1 \begin{vmatrix} a_2 & a_5 \\ a_3 & a_6 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_4 \\ a_3 & a_6 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_4 \\ a_2 & a_5 \end{vmatrix})p_7 \\
&\quad +(a_4 \begin{vmatrix} a_2 & a_5 \\ a_3 & a_6 \end{vmatrix} - a_5 \begin{vmatrix} a_1 & a_4 \\ a_3 & a_6 \end{vmatrix} + a_6 \begin{vmatrix} a_1 & a_4 \\ a_2 & a_5 \end{vmatrix})p_8 \\
&\quad +(a_7 \begin{vmatrix} a_2 & a_5 \\ a_3 & a_6 \end{vmatrix} - a_8 \begin{vmatrix} a_1 & a_4 \\ a_3 & a_6 \end{vmatrix} + a_9 \begin{vmatrix} a_1 & a_4 \\ a_2 & a_5 \end{vmatrix})p_9 \\
&= \begin{vmatrix} a_1 & a_4 & a_1 \\ a_2 & a_5 & a_2 \\ a_3 & a_6 & a_3 \end{vmatrix} p_7 + \begin{vmatrix} a_1 & a_4 & a_4 \\ a_2 & a_5 & a_5 \\ a_3 & a_6 & a_6 \end{vmatrix} p_8 + \begin{vmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{vmatrix} p_9 \\
&= (\det A_{3\times 3})p_9,
\end{aligned}
$$

it follows further that

$$
\begin{aligned}
& \det \left[\frac{\partial R_i}{\partial r_j}\right]_{6\times 6} \\
&= \tfrac{1}{p'_9}((\det A)p_9)(\det B)\{ \begin{vmatrix} b_1 & b_7 \\ b_3 & b_9 \end{vmatrix} + \begin{vmatrix} b_1 & b_7 \\ b_2 & b_8 \end{vmatrix} \tfrac{p'_1p'_8 - p'_2p'_7}{\delta'} + \begin{vmatrix} b_2 & b_8 \\ b_3 & b_9 \end{vmatrix} \tfrac{p'_2p'_9 - p'_3p'_8}{\delta'} \} \\
&= (\det A)(\det A)^2 \tfrac{p_9}{p'_9} \{ \tfrac{1}{\delta'} (a_5(\det A) \begin{vmatrix} p'_1 & p'_7 \\ p'_3 & p'_9 \end{vmatrix} - a_6(\det A) \begin{vmatrix} p'_1 & p'_7 \\ p'_2 & p'_8 \end{vmatrix} - \\
& a_4(\det A) \begin{vmatrix} p'_2 & p'_8 \\ p'_3 & p'_9 \end{vmatrix})\}
\end{aligned}
$$

$$= (\det A)^4 \tfrac{p_9}{p_9'} \tfrac{1}{\delta'} \left(- a_4 \begin{vmatrix} p_2' & p_8' \\ p_3' & p_9' \end{vmatrix} + a_5 \begin{vmatrix} p_1' & p_7' \\ p_3' & p_9' \end{vmatrix} - a_6 \begin{vmatrix} p_1' & p_7' \\ p_2' & p_8' \end{vmatrix} \right)$$

$$= (\det A)^4 \tfrac{p_9}{p_9'} \tfrac{1}{\delta'} \det \begin{bmatrix} p_1' & a_4 & p_7' \\ p_2' & a_5 & p_8' \\ p_3' & a_6 & p_9' \end{bmatrix}$$

$$= (\det A)^4 \tfrac{p_9}{p_9'} \tfrac{1}{\delta'} \det \left(\begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} p_1 & 0 & p_7 \\ p_2 & 1 & p_8 \\ p_3 & 0 & p_9 \end{bmatrix} \right)$$

$$= (\det A)^5 \tfrac{p_9}{p_9'} \tfrac{p_1p_9 - p_3p_7}{p_1'p_9' - p_3'p_7'}$$

$$= \tfrac{(\det P')^5}{(\det P)^5} \tfrac{p_9}{p_9'} \tfrac{p_1p_9 - p_3p_7}{p_1'p_9' - p_3'p_7'}$$

Hence

$$\det \left[\tfrac{\partial(p_1', p_2', \cdots, p_9', R_1, R_3, R_4, R_5, R_6, R_9)}{\partial(p_1, p_2, \cdots, p_9, r_1, r_3, r_4, r_5, r_6, r_9)} \right]$$

$$= \det \left[\tfrac{\partial p_i'}{\partial p_j} \right]_{9\times 9} \det \left[\tfrac{\partial R_i}{\partial r_j} \right]_{6\times 6}$$

$$= \tfrac{(\det P')^3}{(\det P)^3} \tfrac{(\det P')^5}{(\det P)^5} \tfrac{p_9}{p_9'} \tfrac{p_1p_9 - p_3p_7}{p_1'p_9' - p_3'p_7'}$$

$$= \frac{\frac{(\det P')^8}{p_9'(p_1'p_9' - p_3'p_7')}}{\frac{(\det P)^8}{p_9(p_1p_9 - p_3p_7)}}$$

Therefore

$$\frac{|p_9(p_1p_9 - p_3p_7)|}{(\det P)^8} dp_1 dp_2 \cdots dp_9 dr_1 dr_3 dr_4 dr_5 dr_6 dr_9$$

is a left Haar measure on $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})$.

Next step we want to find a right Haar measure.

Given $\mathcal{A}_0 \in C_A$, for each $\mathcal{A} \in C_A$, let $\mathcal{A}' = \mathcal{A}\mathcal{A}_0$. But $\mathcal{A}'$ may not be in the patch C_A. We want to find the corresponding $\tilde{\mathcal{A}} \in \mathrm{Ad}_G \mathcal{A}'$ such that $\tilde{\mathcal{A}}$ is in C_A. Let the matrices of $\mathcal{A}$ and $\mathcal{A}_0$ relative to the basis $\mathcal{B}$ are the same as before. Then

$$[\mathcal{A}']_{\mathcal{B}} = [\mathcal{A}\mathcal{A}_0]_{\mathcal{B}} = \left[\begin{array}{c|c} Q & R \\ \hline 0 & P \end{array}\right] \left[\begin{array}{c|c} B & C \\ \hline 0 & A \end{array}\right] = \left[\begin{array}{c|c} QB & QC+RA \\ \hline 0 & PA \end{array}\right] =: \left[\begin{array}{c|c} Q'_{3\times3} & R'_{3\times3} \\ \hline 0_{3\times3} & P'_{3\times3} \end{array}\right]$$

From previous lemma we know $\tilde{\mathcal{A}}$ has matrix

$$\left[\tilde{\mathcal{A}}\right]_{\mathcal{B}} = \begin{bmatrix} Q'_{3\times3} & \begin{pmatrix} R_1(\vec{P}',\vec{R}') & R_4(\vec{P}',\vec{R}') & 0 \\ 0 & R_5(\vec{P}',\vec{R}') & 0 \\ R_3(\vec{P}',\vec{R}') & R_6(\vec{P}',\vec{R}') & R_9(\vec{P}',\vec{R}') \end{pmatrix} \\ 0_{3\times3} & P'_{3\times3} \end{bmatrix}$$

So we need to know matrices P' and R', then each $R_i(\vec{P}',\vec{R}')$ follows by formula.

$$\begin{aligned} & P' \\ &= PA \\ &= \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \\ &= \begin{bmatrix} a_1p_1 + a_2p_4 + a_3p_7 & a_4p_1 + a_5p_4 + a_6p_7 & a_7p_1 + a_8p_4 + a_9p_7 \\ a_1p_2 + a_2p_5 + a_3p_8 & a_4p_2 + a_5p_5 + a_6p_8 & a_7p_2 + a_8p_5 + a_9p_8 \\ a_1p_3 + a_2p_6 + a_3p_9 & a_4p_3 + a_5p_6 + a_6p_9 & a_7p_3 + a_8p_6 + a_9p_9 \end{bmatrix} \\ &=: \begin{bmatrix} p'_1 & p'_4 & p'_7 \\ p'_2 & p'_5 & p'_8 \\ p'_3 & p'_6 & p'_9 \end{bmatrix} \end{aligned}$$

R'

$= QC + RA$

$$= \begin{bmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{bmatrix} \begin{bmatrix} c_1 & c_4 & 0 \\ 0 & c_5 & 0 \\ c_3 & c_6 & c_9 \end{bmatrix} + \begin{bmatrix} r_1 & r_4 & 0 \\ 0 & r_5 & 0 \\ r_3 & r_6 & r_9 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}$$

$$=: \begin{bmatrix} r'_1 & r'_4 & r'_7 \\ r'_2 & r'_5 & r'_8 \\ r'_3 & r'_6 & r'_9 \end{bmatrix}$$

where

$$r'_1 = a_1r_1 + a_2r_4 + c_1q_1 + c_3q_7$$

$$r'_2 = a_2r_5 + c_1q_2 + c_3q_8$$

$$r'_3 = a_1r_3 + a_2r_6 + a_3r_9 + c_1q_3 + c_3q_9$$

$$r'_4 = a_4r_1 + a_5r_4 + c_4q_1 + c_5q_4 + c_6q_7$$

$$r'_5 = a_5r_5 + c_4q_2 + c_5q_5 + c_6q_8$$

$$r'_6 = a_4r_3 + a_5r_6 + a_6r_9 + c_4q_3 + c_5q_6 + c_6q_9$$

$$r'_7 = a_7r_1 + a_8r_4 + c_9q_7$$

$$r'_8 = a_8r_5 + c_9q_8$$

$$r'_9 = a_7r_3 + a_8r_6 + a_9r_9 + c_9q_9.$$

Let $\delta' := p'_1p'_9 - p'_3p'_7$. By formula in previous lemma, each $R_i(\vec{P}', \vec{R}')$ is the following:

$$R_1(\vec{P}',\vec{R}')$$
$$= r_1' + \frac{p_2'p_9'-p_3'p_8'}{\delta'}r_2' - \frac{p_3'}{p_9'}r_7' - \frac{p_3'(p_2'p_9'-p_3'p_8')}{p_9'\delta'}r_8'$$
$$= (a_1r_1 + a_2r_4 + c_1q_1 + c_3q_7) + \frac{p_2'p_9'-p_3'p_8'}{\delta'}(a_2r_5 + c_1q_2 + c_3q_8)$$
$$-\frac{p_3'}{p_9'}(a_7r_1 + a_8r_4 + c_9q_7) - \frac{p_3'(p_2'p_9'-p_3'p_8')}{p_9'\delta'}(a_8r_5 + c_9q_8)$$
$$= (a_1 - a_7\frac{p_3'}{p_9'})r_1 + (a_2 - a_8\frac{p_3'}{p_9'})r_4 + (a_2\frac{p_2'p_9'-p_3'p_8'}{\delta'} - a_8\frac{p_3'(p_2'p_9'-p_3'p_8')}{p_9'\delta'})r_5 + \text{rational}(\vec{P}),$$
$$R_3(\vec{P}',\vec{R}')$$
$$= \frac{p_1'p_8'-p_2'p_7'}{\delta'}r_2' + r_3' + \frac{p_1'}{p_9'}r_7' + \frac{p_1'(p_2'p_9'-p_3'p_8')}{p_9'\delta'}r_8'$$
$$= \frac{p_1'p_8'-p_2'p_7'}{\delta'}(a_2r_5 + c_1q_2 + c_3q_8) + (a_1r_3 + a_2r_6 + a_3r_9 + c_1q_3 + c_3q_9)$$
$$+\frac{p_1'}{p_9'}(a_7r_1 + a_8r_4 + c_9q_7) + \frac{p_1'(p_2'p_9'-p_3'p_8')}{p_9'\delta'}(a_8r_5 + c_9q_8)$$
$$= (a_7\frac{p_1'}{p_9'})r_1 + a_1r_3 + (a_8\frac{p_1'}{p_9'})r_4 + (a_2\frac{p_1'p_8'-p_2'p_7'}{\delta'} + a_8\frac{p_1'(p_2'p_9'-p_3'p_8')}{p_9'\delta'})r_5$$
$$+a_2r_6 + a_3r_9 + \text{rational}(\vec{P}),$$
$$R_4(\vec{P}',\vec{R}')$$
$$= \frac{p_5'p_9'-p_6'p_8'}{\delta'}r_2' + r_4' - \frac{p_6'}{p_9'}r_7' - \frac{p_3'(p_5'p_9'-p_6'p_8')}{p_9'\delta'}r_8'$$
$$= \frac{p_5'p_9'-p_6'p_8'}{\delta'}(a_2r_5 + c_1q_2 + c_3q_8) + (a_4r_1 + a_5r_4 + c_4q_1 + c_5q_4 + c_6q_7)$$
$$-\frac{p_6'}{p_9'}(a_7r_1 + a_8r_4 + c_9q_7) - \frac{p_3'(p_5'p_9'-p_6'p_8')}{p_9'\delta'}(a_8r_5 + c_9q_8)$$
$$= (a_4 - a_7\frac{p_6'}{p_9'})r_1 + (a_5 - a_8\frac{p_6'}{p_9'})r_4 + (a_2\frac{p_5'p_9'-p_6'p_8'}{\delta'} - a_8\frac{p_3'(p_5'p_9'-p_6'p_8')}{p_9'\delta'})r_5 + \text{rational}(\vec{P}),$$
$$R_5(\vec{P}',\vec{R}')$$
$$= -\frac{p_4'p_9'-p_6'p_7'}{\delta'}r_2' + r_5' - \frac{p_1'p_6'-p_3'p_4'}{\delta'}r_8'$$
$$= -\frac{p_4'p_9'-p_6'p_7'}{\delta'}(a_2r_5 + c_1q_2 + c_3q_8) + (a_5r_5 + c_4q_2 + c_5q_5 + c_6q_8) - \frac{p_1'p_6'-p_3'p_4'}{\delta'}(a_8r_5 + c_9q_8)$$
$$= (-a_2\frac{p_4'p_9'-p_6'p_7'}{\delta'} + a_5 - a_8\frac{p_1'p_6'-p_3'p_4'}{\delta'})r_5 + \text{rational}(\vec{P}),$$
$$R_6(\vec{P}',\vec{R}')$$
$$= \frac{p_4'p_8'-p_5'p_7'}{\delta'}r_2' + r_6' + \frac{p_4'}{p_9'}r_7' + \frac{p_1'p_5'p_9'-p_3'p_4'p_8'}{p_9'\delta'}r_8'$$
$$= \frac{p_4'p_8'-p_5'p_7'}{\delta'}(a_2r_5 + c_1q_2 + c_3q_8) + (a_4r_3 + a_5r_6 + a_6r_9 + c_4q_3 + c_5q_6 + c_6q_9)$$
$$+\frac{p_4'}{p_9'}(a_7r_1 + a_8r_4 + c_9q_7) + \frac{p_1'p_5'p_9'-p_3'p_4'p_8'}{p_9'\delta'}(a_8r_5 + c_9q_8)$$
$$= (a_7\frac{p_4'}{p_9'})r_1 + a_4r_3 + (a_8\frac{p_4'}{p_9'})r_4 + (a_2\frac{p_4'p_8'-p_5'p_7'}{\delta'}) + a_8\frac{p_1'p_5'p_9'-p_3'p_4'p_8'}{p_9'\delta'})r_5$$
$$+a_5r_6 + a_6r_9 + \text{rational}(\vec{P}),$$

$R_9(\vec{P}', \vec{R}')$

$= \frac{p_7'}{p_9'} r_7' + \frac{p_8'}{p_9'} r_8' + r_9'$

$= \frac{p_7'}{p_9'}(a_7 r_1 + a_8 r_4 + c_9 q_7) + \frac{p_8'}{p_9'}(a_8 r_5 + c_9 q_8) + (a_7 r_3 + a_8 r_6 + a_9 r_9 + c_9 q_9)$

$= (a_7 \frac{p_7'}{p_9'}) r_1 + a_7 r_3 + (a_8 \frac{p_7'}{p_9'}) r_4 + (a_8 \frac{p_8'}{p_9'}) r_5 + a_8 r_6 + a_9 r_9 + \text{rational}(\vec{P})$.

Thus we obtain all $R_i(\vec{P}', \vec{R}')$, and hence we get $\tilde{\mathcal{A}} \in \mathrm{Ad}_G \mathcal{A}'$ such that $\tilde{\mathcal{A}}$ is in the patch C_A. Now let's calculate two determinants.

$$\det \left[\frac{\partial p_i'}{\partial p_j}\right]_{9\times 9}$$

$$= \det \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 \\ 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 \\ 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 \\ a_4 & 0 & 0 & a_5 & 0 & 0 & a_6 & 0 & 0 \\ 0 & a_4 & 0 & 0 & a_5 & 0 & 0 & a_6 & 0 \\ 0 & 0 & a_4 & 0 & 0 & a_5 & 0 & 0 & a_6 \\ a_7 & 0 & 0 & a_8 & 0 & 0 & a_9 & 0 & 0 \\ 0 & a_7 & 0 & 0 & a_8 & 0 & 0 & a_9 & 0 \\ 0 & 0 & a_7 & 0 & 0 & a_8 & 0 & 0 & a_9 \end{bmatrix}$$

$$= -\det \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & a_5 & 0 & 0 & a_6 & 0 & 0 \\ a_7 & 0 & 0 & a_8 & 0 & 0 & a_9 & 0 & 0 \\ 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 \\ 0 & a_4 & 0 & 0 & a_5 & 0 & 0 & a_6 & 0 \\ 0 & a_7 & 0 & 0 & a_8 & 0 & 0 & a_9 & 0 \\ 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 \\ 0 & 0 & a_4 & 0 & 0 & a_5 & 0 & 0 & a_6 \\ 0 & 0 & a_7 & 0 & 0 & a_8 & 0 & 0 & a_9 \end{bmatrix}$$

$$
= \det \begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & a_8 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & a_5 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7 & a_8 & a_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_4 & a_5 & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 & a_8 & a_9 \end{bmatrix}
$$

$$
= (\det A)^3
$$

$$
= \frac{(\det P')^3}{(\det P)^3}
$$

$$
\det \left[\frac{\partial R_i}{\partial r_j}\right]_{6\times 6}
$$

$$
= \det \begin{bmatrix} a_1 - a_7\frac{p_3'}{p_9'} & 0 & a_2 - a_8\frac{p_3'}{p_9'} & * & 0 & 0 \\ a_7\frac{p_1'}{p_9'} & a_1 & a_8\frac{p_1'}{p_9'} & * & a_2 & a_3 \\ a_4 - a_7\frac{p_6'}{p_9'} & 0 & a_5 - a_8\frac{p_6'}{p_9'} & * & 0 & 0 \\ 0 & 0 & 0 & -a_2\frac{p_4'p_9'-p_6'p_7'}{\delta'} + a_5 - a_8\frac{p_1'p_6'-p_3'p_4'}{\delta'} & 0 & 0 \\ a_7\frac{p_4'}{p_9'} & a_4 & a_8\frac{p_4'}{p_9'} & * & a_5 & a_6 \\ a_7\frac{p_7'}{p_9'} & a_7 & a_8\frac{p_7'}{p_9'} & * & a_8 & a_9 \end{bmatrix}
$$

$$
= \left(-a_2\frac{p_4'p_9'-p_6'p_7'}{\delta'} + a_5 - a_8\frac{p_1'p_6'-p_3'p_4'}{\delta'}\right) \det \begin{bmatrix} a_1 - a_7\frac{p_3'}{p_9'} & 0 & a_2 - a_8\frac{p_3'}{p_9'} & 0 & 0 \\ a_7\frac{p_1'}{p_9'} & a_1 & a_8\frac{p_1'}{p_9'} & a_2 & a_3 \\ a_4 - a_7\frac{p_6'}{p_9'} & 0 & a_5 - a_8\frac{p_6'}{p_9'} & 0 & 0 \\ a_7\frac{p_4'}{p_9'} & a_4 & a_8\frac{p_4'}{p_9'} & a_5 & a_6 \\ a_7\frac{p_7'}{p_9'} & a_7 & a_8\frac{p_7'}{p_9'} & a_8 & a_9 \end{bmatrix}
$$

$$= \left(-a_2\frac{p_4'p_9'-p_6'p_7'}{\delta'} + a_5 - a_8\frac{p_1'p_6'-p_3'p_4'}{\delta'}\right) \det \begin{bmatrix} a_1 - a_7\frac{p_3'}{p_9'} & a_2 - a_8\frac{p_3'}{p_9'} & 0 & 0 & 0 \\ a_4 - a_7\frac{p_6'}{p_9'} & a_5 - a_8\frac{p_6'}{p_9'} & 0 & 0 & 0 \\ a_7\frac{p_1'}{p_9'} & a_8\frac{p_1'}{p_9'} & a_1 & a_2 & a_3 \\ a_7\frac{p_4'}{p_9'} & a_8\frac{p_4'}{p_9'} & a_4 & a_5 & a_6 \\ a_7\frac{p_7'}{p_9'} & a_8\frac{p_7'}{p_9'} & a_7 & a_8 & a_9 \end{bmatrix}$$

$$= \left(-a_2\frac{p_4'p_9'-p_6'p_7'}{\delta'} + a_5 - a_8\frac{p_1'p_6'-p_3'p_4'}{\delta'}\right)(\det A)$$

$$\cdot\left\{(a_1 - a_7\tfrac{p_3'}{p_9'})(a_5 - a_8\tfrac{p_6'}{p_9'}) - (a_2 - a_8\tfrac{p_3'}{p_9'})(a_4 - a_7\tfrac{p_6'}{p_9'})\right\}$$

$$= (\det A)\frac{1}{\delta'}\left(- a_2 \det \begin{bmatrix} p_4' & p_7' \\ p_6' & p_9' \end{bmatrix} + a_5 \det \begin{bmatrix} p_1' & p_7' \\ p_3' & p_9' \end{bmatrix} - a_8 \det \begin{bmatrix} p_1' & p_4' \\ p_3' & p_6' \end{bmatrix} \right)$$

$$\cdot\left\{(a_1a_5 - a_2a_4) - (a_1a_8 - a_2a_7)\tfrac{p_6'}{p_9'} + (a_4a_8 - a_5a_7)\tfrac{p_3'}{p_9'}\right\}$$

$$= (\det A)\frac{1}{p_9'\delta'} \det \begin{bmatrix} p_1' & p_4' & p_7' \\ a_2 & a_5 & a_8 \\ p_3' & p_6' & p_9' \end{bmatrix} \det \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ p_3' & p_6' & p_9' \end{bmatrix}$$

$$= \frac{\det A}{p_9'(p_1'p_9'-p_3'p_7')} \det\left(\begin{bmatrix} p_1 & p_4 & p_7 \\ 0 & 1 & 0 \\ p_3 & p_6 & p_9 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \right) \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_3 & p_6 & p_9 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \right)$$

$$= (\det A)^3 \frac{p_9(p_1p_9-p_3p_7)}{p_9'(p_1'p_9'-p_3'p_7')}$$

$$= \frac{(\det P')^3 p_9(p_1p_9-p_3p_7)}{(\det P)^3 p_9'(p_1'p_9'-p_3'p_7')}$$

Then

$$\det \left[\frac{\partial(p_1',p_2',\cdots,p_9',R_1,R_3,R_4,R_5,R_6,R_9)}{\partial(p_1,p_2,\cdots,p_9,r_1,r_3,r_4,r_5,r_6,r_9)}\right]$$

$$= \det \left[\frac{\partial p_i'}{\partial p_j}\right]_{9\times 9} \det \left[\frac{\partial R_i}{\partial r_j}\right]_{6\times 6}$$

$$= \frac{(\det P')^3}{(\det P)^3}\frac{(\det P')^3 p_9(p_1p_9-p_3p_7)}{(\det P)^3 p_9'(p_1'p_9'-p_3'p_7')}$$

$$= \frac{\frac{(\det P')^6}{p_9'(p_1'p_9'-p_3'p_7')}}{\frac{(\det P)^6}{p_9(p_1p_9-p_3p_7)}}$$

Therefore

$$\frac{|p_9(p_1p_9 - p_3p_7)|}{(\det P)^6} dp_1 dp_2 \cdots dp_9 dr_1 dr_3 dr_4 dr_5 dr_6 dr_9$$

is a right Haar measure measure on $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})$.

Motivation. From the introductory example for $\mathfrak{g} = \mathcal{F}_{3,2}$, we know that $\sigma \otimes \sigma$ is an automorphism of $\mathfrak{g}$ for all σ in $\mathrm{SO}(3, \mathbf{R})$. We call it the rotation-automorphism.

We now define dilation-automorphisms of $\mathfrak{g}$ by

$$\begin{aligned} \tau_r : \qquad \mathfrak{g}_2 \oplus \mathfrak{g}_1 &\longrightarrow \mathfrak{g}_2 \oplus \mathfrak{g}_1 \\ (Z, Y) &\longmapsto (\tfrac{1}{r} Z, \tfrac{1}{\sqrt{r}} Y) \end{aligned}$$

$\forall r > 0$. Then τ_r indeed is an automorphism of $\mathfrak{g}$ since

$\tau_r([Y_i, Y_j]) = \frac{1}{r}[Y_i, Y_j] = [\frac{1}{\sqrt{r}} Y_i, \frac{1}{\sqrt{r}} Y_j] = [\tau_r Y_i, \tau_r Y_j]$ for all pair (i, j).

And define its dual τ_r^* by the inverse transpose, then

$$\tau_r^*(Z^*, Y^*) = (rZ^*, \sqrt{r} Y^*)$$

since $(\tau_r^*(Z^*, Y^*))(Z', Y') = (Z^*, Y^*)(\tau_r^{-1}(Z', Y')) = (Z^*, Y^*)(rZ', \sqrt{r} Y')$

$= (rZ^*, \sqrt{r} Y^*)(Z', Y')$,$\forall (Z', Y') \in \mathfrak{g}_2 \oplus \mathfrak{g}_1$.

Now let $l_0 \in \mathfrak{g}^*_{Max}$ be fixed, for each $l \in \mathfrak{g}^*_{Max}$, there exist $r > 0$ and $\sigma \in \mathrm{SO}(3)$ such that $r\sigma^*(l_0|_{\mathfrak{g}_2}) = l|_{\mathfrak{g}_2}$ since any two non-zero vectors can be rotated and dilated into one another. Thus we get an automorphism $\tau_r \circ (\sigma \otimes \sigma)$ with $(\tau_r^* \circ (\sigma \otimes \sigma)^*) l_0 = r\sigma^*(l_0|_{\mathfrak{g}_2}) + \sqrt{r}\sigma^*(l_0|_{\mathfrak{g}_1})$, and hence we have the following relations of orbits $\mathcal{O}_{\tau_r^*(\sigma\otimes\sigma)^* l_0} \parallel \mathcal{O}_{r\sigma^*(l_0|_{\mathfrak{g}_2})} = \mathcal{O}_{l|_{\mathfrak{g}_2}} \parallel \mathcal{O}_l$. The purpose of next lemma is to find an automorphism ρ_t^σ associated with σ such that its dual $\rho_t^{\sigma^*}$ moves the orbit $\mathcal{O}_{\tau_r^*(\sigma\otimes\sigma)^* l_0}$ parallel to $\mathcal{O}_l$ along the direction in $\mathfrak{g}_1^*$ related to σ. Then we call ρ_t^σ the translation-automorphism of $\mathfrak{g}$.

Lemma 5.5. For $\mathfrak{g} = \mathcal{F}_{3,2}$, give g a norm respecting the bracket. Let $\mathcal{B} = \{Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\}$ be an orthonormal basis for $\mathfrak{g}$ built as in Construction 2, and let $\mathcal{B}^* = \{Z_1^*, Z_2^*, Z_3^*, Y_1^*, Y_2^*, Y_3^*\}$ be its dual basis in $\mathfrak{g}^*$. Fix $l_0 = a_1 Z_1^* + a_2 Z_2^* + a_3 Z_3^* + b_1 Y_1^* + b_2 Y_2^* + b_3 Y_3^* \in \mathfrak{g}^*_{Max}$, and denote it by $(\vec{a}, \vec{b})$. Choose a cross-section for $\mathrm{SO}(3)/\mathrm{Stab}_{\mathrm{SO}(3)}(\vec{a})$. For each $l \in \mathfrak{g}^*_{Max}$, let $l = z_1 Z_1^* + z_2 Z_2^* + z_3 Z_3^* + y_1 Y_1^* + y_2 Y_2^* + y_3 Y_3^* =: (\vec{z}, \vec{y})$. We show there exists an automorphism ρ_t^σ of $\mathfrak{g}$ with $\sigma \in \Sigma$ such that $\rho_t^{\sigma^*} l = l + t(z_1 Y_1^* + z_2 Y_2^* + z_3 Y_3^*)$, $\forall t \in \mathbf{R}$, where σ^* rotate $\vec{a}$ into the direction of $\vec{z}$. Moreover, let $\vec{u} = \frac{1}{\|\vec{a}\|}(a_1 Y_1^* + a_2 Y_2^* + a_3 Y_3^*)$. For every $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{g}_2^* \oplus \mathfrak{g}_1^*$, $\rho_t^{\sigma^*}(\vec{\alpha}, \vec{\beta}) = (\vec{\alpha}, \vec{\beta} + \frac{t}{\|\vec{a}\|^2}\langle \sigma^* \vec{a}, \vec{\alpha}\rangle \sigma^*(\|\vec{a}\| \vec{u}))$.

Proof. For fixed $l_0 = Z_3^* \in \mathfrak{g}^*_{Max}$, let $\tilde{\rho}_t^e$ be the map

$$\begin{aligned} \tilde{\rho}_t^e : \quad \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Z_i &\longmapsto Z_i, \ i = 1, 2, 3, \\ Y_j &\longmapsto Y_j, \ j = 1, 2, \\ Y_3 &\longmapsto Y_3 - tZ_3 \end{aligned}$$

then $\tilde{\rho}_t^e$ is an automorphism of $\mathfrak{g}$, and its dual $\tilde{\rho}_t^{e^*}$ is

$$\begin{aligned} \tilde{\rho}_t^{e^*} : \quad \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ Y_i^* &\longmapsto Y_i^*, \ i = 1, 2, 3, \\ Z_j^* &\longmapsto Z_j^*, \ j = 1, 2, \\ Z_3^* &\longmapsto Z_3^* + tY_3^* \end{aligned}$$

We verify the dual $\tilde{\rho}_t^{e^*}$ as follows:

$(\tilde{\rho}_t^{e^*}(Y_i^*))(Y_j) = Y_i^*(\tilde{\rho}_t^{e^{-1}} Y_j) = Y_i^*(Y_j), \forall j = 1, 2,$

$(\tilde{\rho}_t^{e^*}(Y_i^*))(Y_3) = Y_i^*(\tilde{\rho}_t^{e^{-1}} Y_3) = Y_i^*(Y_3 + tZ_3) = Y_i^*(Y_3)$

$(\tilde{\rho}_t^{e^*}(Y_i^*))(Z_j) = Y_i^*(\tilde{\rho}_t^{e^{-1}} Z_j) = Y_i^*(Z_j), \forall j = 1, 2, 3.$

So $\tilde{\rho}_t^{e^*} : Y_i^* \longmapsto Y_i^*, \forall i = 1, 2, 3.$

For $i = 1, 2$, we have

$(\tilde{\rho}_t^{e^*}(Z_i^*))(Y_j) = Z_i^*(\tilde{\rho}_t^{e^{-1}} Y_j) = Z_i^*(Y_j), \forall j = 1, 2,$

$(\tilde{\rho}_t^{e^*}(Z_i^*))(Y_3) = Z_i^*(\tilde{\rho}_t^{e^{-1}} Y_3) = Z_i^*(Y_3 + tZ_3) = Z_i^*(Y_3),$

$(\tilde{\rho}_t^{e^*}(Z_i^*))(Z_j) = Z_i^*(\tilde{\rho}_t^{e^{-1}} Z_j) = Z_i^*(Z_j), \forall j = 1, 2, 3.$

So $\tilde{\rho}_t^{e^*} : Z_i^* \longmapsto Z_i^*, \forall i = 1, 2.$

$(\tilde{\rho}_t^{e^*}(Z_3^*))(Y_j) = Z_3^*(\tilde{\rho}_t^{e^{-1}} Y_j) = Z_3^*(Y_j) = (Z_3^* + tY_3^*)(Y_j), \forall j = 1, 2,$

$(\tilde{\rho}_t^{e^*}(Z_3^*))(Y_3) = Z_3^*(\tilde{\rho}_t^{e^{-1}} Y_3) = Z_3^*(Y_3 + tZ_3) = t = (Z_3^* + tY_3^*)(Y_3),$

$(\tilde{\rho}_t^{e^*}(Z_3^*))(Z_j) = Z_3^*(\tilde{\rho}_t^{e^{-1}} Z_j) = Z_3^*(Z_j) = (Z_3^* + tY_3^*)(Z_j), \forall j = 1, 2, 3.$

So $\tilde{\rho}_t^{e^*} : Z_3^* \longmapsto Z_3^* + tY_3^*$. Thus, we get the dual $\tilde{\rho}_t^{e^*}$ as desired.

For each $\sigma \in \mathrm{SO}(3)$, let $\sigma = \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix}$, define

$$\tilde{\rho}_t^{\sigma} = (\sigma \otimes \sigma) \circ \tilde{\rho}_t^{e} \circ (\sigma \otimes \sigma)^{-1}$$

Then we get

$$\begin{aligned} \tilde{\rho}_t^{\sigma} : \quad \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Z_i &\longmapsto Z_i, \ \forall i = 1, 2, 3, \\ Y_1 &\longmapsto Y_1 - tp_7(\sigma Z_3) \\ Y_2 &\longmapsto Y_2 - tp_8(\sigma Z_3) \\ Y_3 &\longmapsto Y_3 - tp_9(\sigma Z_3) \end{aligned}$$

since

$\tilde{\rho}_t^{\sigma}(Y_1) = (\sigma \otimes \sigma)\tilde{\rho}_t^{e}(\sigma^{-1}Y_1) = (\sigma \otimes \sigma)\tilde{\rho}_t^{e}(\sigma^{t}Y_1) = (\sigma \otimes \sigma)\tilde{\rho}_t^{e}(p_1Y_1 + p_4Y_2 + p_7Y_3) =$

$\sigma(p_1Y_1 + p_4Y_2 + p_7Y_3) - tp_7(\sigma Z_3) = Y_1 - tp_7(\sigma Z_3),$

and it's similar for Y_2 and Y_3. We also get the dual

$\tilde{\rho}_t^{\sigma^*} = (\sigma \otimes \sigma)^* \circ \tilde{\rho}_t^{e^*} \circ (\sigma^{-1} \otimes \sigma^{-1})^*,$

$$\begin{aligned} \tilde{\rho}_t^{\sigma^*} : \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ Y_i^* &\longmapsto Y_i^*, \ \forall i = 1,2,3, \\ Z_1^* &\longmapsto Z_1^* + tp_7(\sigma^* Y_3^*) \\ Z_2^* &\longmapsto Z_2^* + tp_8(\sigma^* Y_3^*) \\ Z_3^* &\longmapsto Z_3^* + tp_9(\sigma^* Y_3^*) \end{aligned}$$

since

$\tilde{\rho}_t^{\sigma^*}(Z_1^*) = (\sigma \otimes \sigma)^* \tilde{\rho}_t^{e^*}(\sigma^{-1^*} Z_1^*) = (\sigma \otimes \sigma)^* \tilde{\rho}_t^{e^*}(p_1Z_1^* + p_4Z_2^* + p_7Z_3^*) =$

$\sigma^*(p_1Z_1^* + p_4Z_2^* + p_7Z_3^*) + tp_7(\sigma^* Y_3^*) = Z_1^* + tp_7(\sigma^* Y_3^*)$,

and it's similar for Z_2^* and Z_3^*.

Since $l_0 = (\vec{a}, \vec{b}) \in \mathfrak{g}_{Max}^*$, $\vec{a} \neq 0$. Then there exists unique $\sigma_a \in \Sigma$ such that $\sigma_a^*(\frac{1}{\|\vec{a}\|}\vec{a}) = Z_3^*$, i.e., $\|\vec{a}\|\sigma_a^{-1^*} Z_3^* = \vec{a}$.

So we have

$\|\vec{a}\|\sigma_a^{-1^*} Y_3^* = a_1Y_1^* + a_2Y_2^* + a_3Y_3^*$, and $\tilde{\rho}_t^{\sigma_a^{-1}} = (\sigma_a^{-1} \otimes \sigma_a^{-1})\tilde{\rho}_t^{e}(\sigma_a \otimes \sigma_a)$.

Hence

$$\begin{aligned} \tilde{\rho}_t^{\sigma_a^{-1^*}} \vec{a} &= (\sigma_a^{-1} \otimes \sigma_a^{-1})^* \tilde{\rho}_t^{e^*}(\sigma_a \otimes \sigma_a)^*(\|\vec{a}\|\sigma_a^{-1^*} Z_3^*) \\ &= \|\vec{a}\|(\sigma_a^{-1} \otimes \sigma_a^{-1})^* \tilde{\rho}_t^{e^*} Z_3^* \\ &= \|\vec{a}\|(\sigma_a^{-1^*} Z_3^* + t\sigma_a^{-1^*} Y_3^*) \\ &= \vec{a} + t(a_1Y_1^* + a_2Y_2^* + a_3Y_3^*) \end{aligned}$$

For each $l = (\vec{z}, \vec{y}) \in \mathfrak{g}^*_{Max}$, $\vec{z} = l|_{\mathfrak{g}_2} \neq 0$. Then there exists unique $(r, \sigma) \in \mathbf{R}^+ \times \Sigma$ such that $\vec{z} = r\sigma^* \vec{a}$.

So we have

$z_1 Z_1^* + z_2 Z_2^* + z_3 Z_3^* = r\sigma^*(a_1 Z_1^* + a_2 Z_2^* + a_3 Z_3^*)$

and

$z_1 Y_1^* + z_2 Y_2^* + z_3 Y_3^* = r\sigma^*(a_1 Y_1^* + a_2 Y_2^* + a_3 Y_3^*)$.

Now we define

$$\rho_t^\sigma = (\sigma \otimes \sigma)\tilde{\rho}_t^{\sigma_a^{-1}}(\sigma \otimes \sigma)^{-1}$$

Then

$$\begin{aligned}
\rho_t^{\sigma^*} l &= \rho_t^{\sigma^*}(\vec{z} + \vec{y}) \\
&= \rho_t^{\sigma^*}(r\sigma^* \vec{a} + \vec{y}) \\
&= r\rho_t^{\sigma^*}(\sigma^* \vec{a}) + \rho_t^{\sigma^*}(\vec{y}) \\
&= r(\sigma \otimes \sigma)^* \tilde{\rho}_t^{\sigma_a^{-1*}} (\sigma \otimes \sigma)^{-1^*}(\sigma^* \vec{a}) + \vec{y} \\
&= r(\sigma \otimes \sigma)^* \tilde{\rho}_t^{\sigma_a^{-1*}} (\vec{a}) + \vec{y} \\
&= r(\sigma \otimes \sigma)^*(\vec{a} + t(a_1 Y_1^* + a_2 Y_2^* + a_3 Y_3^*)) + \vec{y} \\
&= (r\sigma^* \vec{a} + \vec{y}) + tr\sigma^*(a_1 Y_1^* + a_2 Y_2^* + a_3 Y_3^*) \\
&= l + t(z_1 Y_1^* + z_2 Y_2^* + z_3 Y_3^*)
\end{aligned}$$

Since

$$\begin{aligned}
\rho_t^\sigma &= (\sigma \otimes \sigma)\tilde{\rho}_t^{\sigma_a^{-1}}(\sigma \otimes \sigma)^{-1} \\
&= (\sigma \otimes \sigma)(\sigma_a^{-1} \otimes \sigma_a^{-1})\tilde{\rho}_t^e(\sigma_a \otimes \sigma_a)(\sigma \otimes \sigma)^{-1} \\
&= (\sigma\sigma_a^{-1} \otimes \sigma\sigma_a^{-1})\tilde{\rho}_t^e(\sigma\sigma_a^{-1} \otimes \sigma\sigma_a^{-1})^{-1}
\end{aligned}$$

let $\sigma\sigma_a^{-1} = \begin{bmatrix} p_1 & p_4 & p_7 \\ p_2 & p_5 & p_8 \\ p_3 & p_6 & p_9 \end{bmatrix}$, then we get $\rho_t^{\sigma^*}(Z_1^*) = Z_1^* + tp_7(\sigma\sigma_a^{-1})^* Y_3^*$,

$\rho_t^{\sigma^*}(Z_2^*) = Z_2^* + tp_8(\sigma\sigma_a^{-1})^* Y_3^*$, and $\rho_t^{\sigma^*}(Z_3^*) = Z_3^* + tp_9(\sigma\sigma_a^{-1})^* Y_3^*$.

Hence for every $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{g}_2^* \oplus \mathfrak{g}_1^*$, let

$\vec{\alpha} = \alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^*$, and $\vec{\beta} = \beta_1 Y_1^* + \beta_2 Y_2^* + \beta_3 Y_3^*$.

Then

$$\begin{aligned}
\rho_t^{\sigma^*}(\vec{\alpha}, \vec{\beta}) &= \alpha_1 \rho_t^{\sigma^*}(Z_1^*) + \alpha_2 \rho_t^{\sigma^*}(Z_2^*) + \alpha_3 \rho_t^{\sigma^*}(Z_3^*) + \rho_t^{\sigma^*}(\vec{\beta}) \\
&= \alpha_1 Z_1^* + \alpha_2 Z_2^* + \alpha_3 Z_3^* + t(p_7\alpha_1 + p_8\alpha_2 + p_9\alpha_3)(\sigma\sigma_a^{-1})^* Y_3^* + \vec{\beta} \\
&= (\vec{\alpha}, \vec{\beta} + t(p_7\alpha_1 + p_8\alpha_2 + p_9\alpha_3)(\sigma\sigma_a^{-1})^* Y_3^*) \\
&= (\vec{\alpha}, \vec{\beta} + t\langle \sigma^* \sigma_a^{-1^*} Z_3^*, \vec{\alpha}\rangle \sigma^*(\frac{1}{||\vec{a}||}(a_1 Y_1^* + a_2 Y_2^* + a_3 Y_3^*))) \\
&= (\vec{\alpha}, \vec{\beta} + t\langle \sigma^*(\frac{1}{||\vec{a}||}\vec{a}), \vec{\alpha}\rangle \sigma^* \vec{u}) \\
&= (\vec{\alpha}, \vec{\beta} + \frac{t}{||\vec{a}||^2}\langle \sigma^* \vec{a}, \vec{\alpha}\rangle \sigma^*(||\vec{a}||\vec{u}))
\end{aligned}$$

This proves the lemma.

Lemma 5.6. Fix $l_0 = a_1 Z_1^* + a_2 Z_2^* + a_3 Z_3^* + b_1 Y_1^* + b_2 Y_2^* + b_3 Y_3^* \in \mathfrak{g}_{Max}^*$, and denote it by $(\vec{a}, \vec{b})$. Let Σ be a cross-section of $SO(3)/\mathrm{Stab}_{SO(3)}(\vec{a})$. Show

$$\mathbf{X} := \{\rho_t^\sigma \circ \tau_r \circ (\sigma \otimes \sigma) |\ (t, r, \sigma) \in \mathbf{R} \times \mathbf{R}^+ \times \Sigma\}$$

is a cross-section of the double coset space $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})/\mathrm{Stab}^*(l_0)$.

Proof. Since fixed $l_0 = (\vec{a}, \vec{b}) \in \mathfrak{g}_{Max}^*$, there is some coefficient $a_i \neq 0$, $1 \leq i \leq 3$. Without loss of generality we let $a_3 \neq 0$. Then claim

$\mathfrak{r}_{l_0} = \mathfrak{g}_2 \oplus \mathbf{R}(a_1 Y_1 + a_2 Y_2 + a_3 Y_3)$.

Since $\mathfrak{r}_{l_0} = \{Z + Y |\ l_0([Z + Y, \mathfrak{g}]) = 0\}$, let $Y = y_1 Y_1 + y_2 Y_2 + y_3 Y_3$, then it follows $l_0([y_1 Y_1 + y_2 Y_2 + y_3 Y_3, Y_i]) = 0$, $\forall i = 1, 2, 3$. If $i = 1$, then $-a_3 y_2 + a_2 y_3 = 0$; if $i = 2$, then $a_3 y_1 - a_1 y_3 = 0$; if $i = 3$, then $-a_2 y_1 + a_1 y_2 = 0$. Since $a_3 \neq 0$, we get $y_1 = \frac{a_1}{a_3} y_3$ and $y_2 = \frac{a_2}{a_3} y_3$, so

$y_1Y_1 + y_2Y_2 + y_3Y_3 = \frac{a_1}{a_3}y_3Y_1 + \frac{a_2}{a_3}y_3Y_2 + y_3Y_3 = \frac{y_3}{a_3}(a_1Y_1 + a_2Y_2 + a_3Y_3)$. Hence $\mathfrak{r}_{l_0} = \{Z + \frac{y_3}{a_3}(a_1Y_1 + a_2Y_2 + a_3Y_3) \mid Z \in \mathfrak{g}_2,\ y_3 \in \mathbf{R}\} = \mathfrak{g}_2 \oplus \mathbf{R}(a_1Y_1 + a_2Y_2 + a_3Y_3)$, and we see $\mathfrak{r}_{l_0}$ is an ideal in $\mathfrak{g}$. Then by Theorem 3.2.3. in [1], we know $\mathcal{O}_{l_0} = l_0 + \mathfrak{r}_{l_0}^{\perp}$ where the annihilator of $\mathfrak{r}_{l_0}$,
$\mathfrak{r}_{l_0}^{\perp} = \mathfrak{r}_{\vec{a}}^{\perp} = Sp\{a_3Y_1^* - a_1Y_3^*, a_3Y_2^* - a_2Y_3^*\}$, is a two-dimensional plane in $\mathfrak{g}_1^*$.
Hence we can choose a unit vector $\vec{u} = \frac{1}{\|\vec{a}\|}(a_1Y_1^* + a_2Y_2^* + a_3Y_3^*) \in \mathfrak{g}_1^*$ perpendicular to this plane $\mathfrak{r}_{\vec{a}}^{\perp}$. Before we go further, let's verify one fact.
For each $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ and $l \in \mathfrak{g}^*$, claim $\mathfrak{r}_{\mathcal{A}^* l}^{\perp} = \mathcal{A}^*(\mathfrak{r}_l^{\perp})$.
Proof of claim:

$$\begin{aligned}
\mathfrak{r}_{\mathcal{A}^* l}^{\perp} &= \{f \in \mathfrak{g}^* \mid f(\mathfrak{r}_{\mathcal{A}^* l}) = 0\} \\
&= \{f \in \mathfrak{g}^* \mid f(\mathcal{A}\mathfrak{r}_l) = 0\} \\
&= \{f = \tilde{f} \circ \mathcal{A}^{-1} \mid \tilde{f}(\mathfrak{r}_l) = 0\} \\
&= \{f = \mathcal{A}^* \tilde{f} \mid \tilde{f}(\mathfrak{r}_l) = 0\} \\
&= \mathcal{A}^*\{\tilde{f} \mid \tilde{f}(\mathfrak{r}_l) = 0\} \\
&= \mathcal{A}^* \mathfrak{r}_l^{\perp}
\end{aligned}$$

For each $\sigma \in \Sigma$, since $(\sigma \otimes \sigma) \in \mathrm{Aut}(\mathfrak{g})$, we get $\sigma^* \mathfrak{r}_{\vec{a}}^{\perp} = (\sigma \otimes \sigma)^* \mathfrak{r}_{\vec{a}}^{\perp} = \mathfrak{r}_{\sigma^* \vec{a}}^{\perp}$.
Since fixed $l_0 = (\vec{a}, \vec{b}) \in \mathfrak{g}_{Max}^*$, $\vec{a} = l_0|_{\mathfrak{g}_2} \neq \vec{0}$. For each $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, we know $\mathcal{A}^* l_0 \in \mathfrak{g}_{Max}^*$, so $(\mathcal{A}^* l_0)|_{\mathfrak{g}_2} \neq 0$. Then there exists unique $(r, \sigma) \in \mathbf{R}^+ \times \Sigma$ such that $r\sigma^* \vec{a} = (\mathcal{A}^* l_0)|_{\mathfrak{g}_2}$. Thus we get unique automorphism $\tau_r \circ (\sigma \otimes \sigma)$ with $\tau_r^*(\sigma \otimes \sigma)^* l_0 = \tau_r^*(\sigma^* \vec{a} + \sigma^* \vec{b}) = r\sigma^* \vec{a} + \sqrt{r}\sigma^* \vec{b}$, and its orbit

$$\mathcal{O}_{\tau_r^*(\sigma\otimes\sigma)^* l_0} = (r\sigma^* \vec{a} + \sqrt{r}\sigma^* \vec{b}) + \mathfrak{r}_{(r\sigma^* \vec{a} + \sqrt{r}\sigma^* \vec{b})}^{\perp} = r\sigma^* \vec{a} + \sqrt{r}\sigma^* \vec{b} + \mathfrak{r}_{\sigma^* \vec{a}}^{\perp}$$

Let $z_1Z_1^* + z_2Z_2^* + z_3Z_3^* = \vec{z} := \sigma^* \vec{a}$, and $y_1Y_1^* + y_2Y_2^* + y_3Y_3^* = \vec{y} := \sigma^* \vec{b}$. Then we have $z_1Y_1^* + z_2Y_2^* + z_3Y_3^* = \sigma^*(\|\vec{a}\|\vec{u})$. Since σ^* rotate $\vec{a}$ into the direction of $\vec{z}$, by previous lemma we know there exists $\rho_t^{\sigma} \in \mathrm{Aut}(\mathfrak{g})$ such that

$\rho_t^{\sigma^*}(\tau_r^*(\sigma\otimes\sigma)^* l_0) = \rho_t^{\sigma^*}(r\vec{z}+\sqrt{r}\vec{y}) = (r\vec{z}+\sqrt{r}\vec{y}) + t(rz_1Y_1^* + rz_2Y_2^* + rz_3Y_3^*) = r\sigma^*\vec{a} + \sqrt{r}\sigma^*\vec{b} + t(r\sigma^*(\|\vec{a}\|\vec{u}))$, and its orbit is parallel to the orbit of $\tau_r^*(\sigma\otimes\sigma)^* l_0$, in other words,

$$\begin{aligned}
\mathcal{O}_{\rho_t^{\sigma^*}\tau_r^*(\sigma\otimes\sigma)^* l_0} &= r\sigma^*\vec{a} + \sqrt{r}\sigma^*\vec{b} + t(r\sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{\rho_t^{\sigma^*}\tau_r^*(\sigma\otimes\sigma)^* l_0} \\
&= r\sigma^*\vec{a} + \sqrt{r}\sigma^*\vec{b} + t(r\sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{\sigma^*\vec{a}} \\
&\parallel r\sigma^*\vec{a} + \sqrt{r}\sigma^*\vec{b} + \mathfrak{r}^{\perp}_{\sigma^*\vec{a}} \\
&= \mathcal{O}_{\tau_r^*(\sigma\otimes\sigma)^* l_0}
\end{aligned}$$

Thus, for each $t \in \mathbf{R}$ we have the following relations of orbits

$$\mathcal{O}_{\rho_t^{\sigma^*}\tau_r^*(\sigma\otimes\sigma)^* l_0} \parallel \mathcal{O}_{\tau_r^*(\sigma\otimes\sigma)^* l_0} = \mathcal{O}_{r\sigma^*\vec{a}+\sqrt{r}\sigma^*\vec{b}} \parallel \mathcal{O}_{r\sigma^*\vec{a}} = \mathcal{O}_{(\mathcal{A}^* l_0)|_{\mathfrak{g}_2}} \parallel \mathcal{O}_{\mathcal{A}^* l_0}$$

We want to show that there exists unique $t \in \mathbf{R}$ such that $\mathcal{O}_{\rho_t^{\sigma^*}\tau_r^*(\sigma\otimes\sigma)^* l_0} = \mathcal{O}_{\mathcal{A}^* l_0}$. Since $\vec{u} = \frac{1}{\|\vec{a}\|}(a_1Y_1^* + a_2Y_2^* + a_3Y_3^*) \notin \mathfrak{r}^{\perp}_{l_0} = \mathfrak{r}^{\perp}_{\vec{a}}$ in $\mathfrak{g}_1^*$, for each $t \in \mathbf{R}$ we have $(tr\|\vec{a}\|\vec{u} + \sqrt{r}\vec{b} + \mathfrak{r}^{\perp}_{\vec{a}}) \parallel \mathfrak{r}^{\perp}_{\vec{a}}$, and hence $\sigma^*(tr\|\vec{a}\|\vec{u} + \sqrt{r}\vec{b} + \mathfrak{r}^{\perp}_{\vec{a}}) \parallel \sigma^*\mathfrak{r}^{\perp}_{\vec{a}}$ in $\mathfrak{g}_1^*$. So

$$\begin{aligned}
\mathfrak{g}_1^* &= \dot{\cup}\, \{\sigma^*(tr\|\vec{a}\|\vec{u} + \sqrt{r}\vec{b} + \mathfrak{r}^{\perp}_{\vec{a}}) \mid t \in \mathbf{R}\} \\
&= \dot{\cup}\, \{t(r\sigma^*(\|\vec{a}\|\vec{u}) + \sqrt{r}\sigma^*\vec{b} + \sigma^*\mathfrak{r}^{\perp}_{\vec{a}} \mid t \in \mathbf{R}\} \\
&= \dot{\cup}\, \{\sqrt{r}\sigma^*\vec{b} + t(r\sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{\sigma^*\vec{a}} \mid t \in \mathbf{R}\}
\end{aligned}$$

Hence there exists <u>*unique*</u> $t \in \mathbf{R}$ such that $(\mathcal{A}^* l_0)|_{\mathfrak{g}_1} \in (\sqrt{r}\sigma^*\vec{b} + t(r\sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{\sigma^*\vec{a}})$. Then

$$\begin{aligned}
\mathcal{A}^* l_0 &= (\mathcal{A}^* l_0)|_{\mathfrak{g}_2} + (\mathcal{A}^* l_0)|_{\mathfrak{g}_1} \\
&= r\sigma^*\vec{a} + (\mathcal{A}^* l_0)|_{\mathfrak{g}_1} \\
&\in r\sigma^*\vec{a} + \sqrt{r}\sigma^*\vec{b} + t(r\sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{\sigma^*\vec{a}} \\
&= \mathcal{O}_{\rho_t^{\sigma^*}\tau_r^*(\sigma\otimes\sigma)^* l_0}
\end{aligned}$$

Hence we get $\mathcal{O}_{\mathcal{A}^* l_0} = \mathcal{O}_{\rho_t^{\sigma^*}\tau_r^*(\sigma\otimes\sigma)^* l_0}$ as desired.

Thus we know for each $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ there exists unique $(t, r, \sigma) \in \mathbf{R} \times \mathbf{R}^+ \times \Sigma$ such that $\mathcal{O}_{\mathcal{A}^* l_0} = \mathcal{O}_{(\rho_t^\sigma \tau_r(\sigma \otimes \sigma))^* l_0}$. And by Theorem 2.1. we know for each $\mathcal{O}_l \in \mathcal{O}_{Max}$ there exists $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathcal{O}_l = \mathcal{O}_{\mathcal{A}^* l_0}$. Now define $T : \mathbf{X}^* \longrightarrow \mathcal{O}_{Max}$ by $T(x^*) = \mathcal{O}_{x^* l_0}$, $\forall x \in \mathbf{X}$. Then T is bijective. Since there is one-to-one correspondence between $\mathcal{O}_{Max}$ and $D := \mathrm{Ad}_G^* \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(l_0)$, it follows that there exists uniquely one $x^* \in \mathbf{X}^*$ from each coset $d \in D$. It concludes that $\mathbf{X}^*$ is a cross-section of D, in other words, $\mathbf{X}$ is a cross-section of $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})/\mathrm{Stab}^*(l_0)$.

Motivation. Let $\mathbf{X} = \{\rho_t^\sigma \circ \tau_r \circ (\sigma \otimes \sigma) |\ \forall (t, r, \sigma) \in \mathbf{R} \times \mathbf{R}^+ \times \Sigma\}$, we have shown that $\mathbf{X}$ is a cross-section of $\mathrm{Ad}_G \backslash \mathrm{Aut}(\mathfrak{g})/\mathrm{Stab}^*(l_0)$. But $\mathbf{X}$ is not a group, i.e., it is not closed under composition. We show next to which element of $\mathbf{X}$ the composition of two elements of $\mathbf{X}$ corresponds.

Lemma 5.7. Fix $l_0 \in \mathfrak{g}^*_{Max}$. Let $l_0 = a_1 Z_1^* + a_2 Z_2^* + a_3 Z_3^* + b_1 Y_1^* + b_2 Y_2^* + b_3 Y_3^*$, and denote it by $(\vec{a}, \vec{b}) \in \mathfrak{g}_2^* \oplus \mathfrak{g}_1^*$. For each $\rho_{t_0}^{\sigma_0} \tau_{r_0}(\sigma_0 \otimes \sigma_0)$, $\rho_t^\sigma \tau_r(\sigma \otimes \sigma) \in \mathbf{X}$, we have the equality

$$\mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0} \tau_{r_0}(\sigma_0 \otimes \sigma_0) \rho_t^\sigma \tau_r(\sigma \otimes \sigma))^* l_0 = \mathrm{Ad}_G^*(\rho^{\sigma_0\sigma}_{\frac{t}{\sqrt{r_0}} + t_0 \cos^2\psi} \tau_{r_0 r}(\sigma_0\sigma \otimes \sigma_0\sigma))^* l_0$$

where ψ is the angle between vectors $\vec{a}$ and $\sigma^* \vec{a}$, i.e., ψ is the angle of rotation for σ^*.

Proof. For fixed $l_0 = (\vec{a}, \vec{b})$, we showed already in the beginning of Lemma5.6, its radical is $\mathfrak{r}_{l_0} = \mathfrak{g}_2 \oplus \mathbf{R}(a_1 Y_1 + a_2 Y_2 + a_3 Y_3)$. So $\dim(\mathfrak{r}_{l_0}) = 4$, and hence $\dim(\mathfrak{r}_{l_0}^\perp) = 2$. Then $\mathfrak{r}_{l_0}^\perp = \mathrm{Span}_{\mathbf{R}}\{a_3 Y_1^* - a_1 Y_3^*, a_3 Y_2^* - a_2 Y_3^*\}$ since it contains the two listed independent vectors.

Now we choose a unit vector $\vec{u} \in \mathfrak{g}_1^*$ perpendicular to the plane $\mathfrak{r}_{l_0}^{\perp}$,

$$\vec{u} = \frac{1}{\|\vec{a}\|}(a_1Y_1^* + a_2Y_2^* + a_3Y_3^*)$$

For each $(\vec{\alpha}, \vec{\beta}) \in \mathfrak{g}_2^* \oplus \mathfrak{g}_1^*$, let's recall that $\rho_t^{\sigma^*}(\vec{\alpha}, \vec{\beta}) = (\vec{\alpha}, \vec{\beta} + \frac{t}{\|\vec{a}\|^2}\langle\sigma^*\vec{a}, \vec{\alpha}\rangle\sigma^*(\|\vec{a}\|\vec{u})$ and $\mathrm{Ad}_G^*(\vec{\alpha}, \vec{\beta}) = (\vec{\alpha}, \vec{\beta}) + \mathfrak{r}_{\vec{a}}^{\perp}$. Then for each $\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)$, $\sigma_t^{\sigma}\tau_r(\sigma \otimes \sigma) \in \mathbf{X}$, the orbit of the dual of their composition acting on l_0 is

$$
\begin{aligned}
& \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma))^*(\vec{a}, \vec{b}) \\
= \ & \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r)^*(\sigma^*\vec{a}, \sigma^*\vec{b}) \\
= \ & \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma})^*(r\sigma^*\vec{a}, \sqrt{r}\sigma^*\vec{b}) \\
= \ & \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0))^*(r\sigma^*\vec{a}, \sqrt{r}\sigma^*\vec{b} + \frac{t}{\|\vec{a}\|^2}\langle\sigma^*\vec{a}, r\sigma^*\vec{a}\rangle\sigma^*(\|\vec{a}\|\vec{u})) \\
= \ & \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0))^*(r\sigma^*\vec{a}, \sqrt{r}\sigma^*\vec{b} + tr\sigma^*(\|\vec{a}\|\vec{u})) \\
= \ & \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0})^*(r\sigma_0^*\sigma^*\vec{a}, \sqrt{r}\sigma_0^*\sigma^*\vec{b} + tr\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u})) \\
= \ & \mathrm{Ad}_G^*\rho_{t_0}^{\sigma_0^*}(r_0r\sigma_0^*\sigma^*\vec{a}, \sqrt{r_0r}\sigma_0^*\sigma^*\vec{b} + t\sqrt{r_0}r\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u})) \\
= \ & \mathrm{Ad}_G^*\rho_{t_0}^{\sigma_0^*}(r_0r\sigma_0^*\sigma^*\vec{a}, \sqrt{r_0r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}}r_0r\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u})) \\
= \ & \mathrm{Ad}_G^*(r_0r\sigma_0^*\sigma^*\vec{a}, \sqrt{r_0r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}}r_0r\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u}) + \frac{t_0}{\|\vec{a}\|^2}\langle\sigma_0^*\vec{a}, r_0r\sigma_0^*\sigma^*\vec{a}\rangle\sigma_0^*(\|\vec{a}\|\vec{u})) \\
= \ & \mathrm{Ad}_G^*(r_0r\sigma_0^*\sigma^*\vec{a}, \sqrt{r_0r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}}r_0r\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u}) + \frac{t_0}{\|\vec{a}\|^2}r_0r\langle\vec{a}, \sigma^*\vec{a}\rangle\sigma_0^*(\|\vec{a}\|\vec{u}))
\end{aligned}
$$

Let ψ be the angle between vectors $\vec{a}$ and $\sigma^*\vec{a}$, then it follows

$$
\begin{aligned}
& \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma))^*(\vec{a}, \vec{b}) \\
= \ & \mathrm{Ad}_G^*(r_0r\sigma_0^*\sigma^*\vec{a}, \sqrt{r_0r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}}r_0r\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u}) + t_0(cos\psi)r_0r\sigma_0^*(\|\vec{a}\|\vec{u})) \\
= \ & (r_0r\sigma_0^*\sigma^*\vec{a}, \sqrt{r_0r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}}r_0r\sigma_0^*\sigma^*(\|\vec{a}\|\vec{u}) + t_0(cos\psi)r_0r\sigma_0^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}_{r_0r\sigma_0^*\sigma^*\vec{a}}^{\perp}
\end{aligned}
$$

Since unit vector $\vec{u} = \frac{1}{\|\vec{a}\|}(a_1Y_1^* + a_2Y_2^* + a_3Y_3^*)$ perpendicular to two-dimensional plane $\mathfrak{r}_{l_0}^{\perp} = \mathfrak{r}_{\vec{a}}^{\perp}$ in $\mathfrak{g}_1^* \cong \mathbf{R}^3$, it implies that the unit vector $\sigma_0^*\sigma^*\vec{u}$ perpendicular to two-dimensional plane $\sigma_0^*\sigma^*\mathfrak{r}_{\vec{a}}^{\perp} = \mathfrak{r}_{\sigma_0^*\sigma^*\vec{a}}^{\perp} = \mathfrak{r}_{r_0r\sigma_0^*\sigma^*\vec{a}}^{\perp}$ in $\mathfrak{g}_1^*$. Then the vector

$\sigma_0^*(\|\vec{a}\|\vec{u}) \in \mathfrak{g}_1^*$ can be written as the addition $\langle \sigma_0^*(\|\vec{a}\|\vec{u}), \sigma_0^*\sigma^*\vec{u}\rangle \sigma_0^*\sigma^*\vec{u} + \vec{v}$ for some vector $\vec{v}$ in the plane $\mathfrak{r}^{\perp}_{r_0 r \sigma_0^* \sigma^* \vec{a}}$. And hence it follows that

$$\begin{aligned}
& \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma))^*(\vec{a}, \vec{b}) \\
= & (r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}} r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u}) \\
& + t_0(cos\psi) r_0 r(\langle \sigma_0^*(\|\vec{a}\|\vec{u}), \sigma_0^*\sigma^*\vec{u}\rangle \sigma_0^*\sigma^*\vec{u} + \vec{v})) + \mathfrak{r}^{\perp}_{r_0 r \sigma_0^* \sigma^* \vec{a}} \\
= & (r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}} r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u}) \\
& + t_0(cos\psi) r_0 r \|\vec{a}\| \langle \vec{u}, \sigma^*\vec{u}\rangle \sigma_0^*\sigma^*\vec{u} + t_0(cos\psi) r_0 r \vec{v}) + \mathfrak{r}^{\perp}_{r_0 r \sigma_0^* \sigma^* \vec{a}} \\
= & (r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}} r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u}) + t_0(cos\psi) r_0 r \langle \vec{u}, \sigma^*\vec{u}\rangle \sigma_0^*\sigma^*(\|\vec{a}\|\vec{u})) \\
& + \mathfrak{r}^{\perp}_{r_0 r \sigma_0^* \sigma^* \vec{a}} \\
= & (r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + \frac{t}{\sqrt{r_0}} r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u}) + t_0(cos\psi)^2 r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{r_0 r \sigma_0^* \sigma^* \vec{a}}
\end{aligned}$$

The last equality holds because ψ is the angle between $\vec{a}$ and $\sigma^*\vec{a}$, meanwhile, it is the angle between $\vec{u}$ and $\sigma^*\vec{u}$. Furthermore, it follows

$$\begin{aligned}
& \mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma))^*(\vec{a}, \vec{b}) \\
= & (r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + (\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi) r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u})) + \mathfrak{r}^{\perp}_{r_0 r \sigma_0^* \sigma^* \vec{a}} \\
= & \mathrm{Ad}_G^*(r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + (\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi) r_0 r \sigma_0^* \sigma^*(\|\vec{a}\|\vec{u})) \\
= & \mathrm{Ad}_G^*(r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b} + \frac{\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi}{\|\vec{a}\|^2} \langle \sigma_0^*\sigma^*\vec{a}, r_0 r \sigma_0^*\sigma^*\vec{a}\rangle \sigma_0^*\sigma^*(\|\vec{a}\|\vec{u})) \\
= & \mathrm{Ad}_G^* \rho^{\sigma_0 \sigma^*}_{\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi}(r_0 r \sigma_0^* \sigma^* \vec{a}, \sqrt{r_0 r}\sigma_0^*\sigma^*\vec{b}) \\
= & \mathrm{Ad}_G^*(\rho^{\sigma_0 \sigma}_{\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi}\tau_{r_0 r})^*(\sigma_0^*\sigma^*\vec{a}, \sigma_0^*\sigma^*\vec{b}) \\
= & \mathrm{Ad}_G^*(\rho^{\sigma_0 \sigma}_{\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi}\tau_{r_0 r}(\sigma_0\sigma \otimes \sigma_0\sigma))^*(\vec{a}, \vec{b})
\end{aligned}$$

Therefore we get the desired equality

$$\mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma))^*(\vec{a}, \vec{b}) = \mathrm{Ad}_G^*(\rho^{\sigma_0 \sigma}_{\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi}\tau_{r_0 r}(\sigma_0\sigma \otimes \sigma_0\sigma))^*(\vec{a}, \vec{b})$$

for every $\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)$, $\rho_t^{\sigma}\tau(\sigma \otimes \sigma) \in \mathbf{X}$.

Verification of Relative Invariance for $\mathcal{F}_{3,2}$

Let $\mathfrak{g} = \mathcal{F}_{3,2}$, choose bases $\mathcal{B}$ for $\mathfrak{g}$ and $\mathcal{B}^*$ for $\mathfrak{g}^*$ just like in Construction 2. Define a linear map $A : \mathfrak{g}_2^* \longrightarrow \mathfrak{g}_1^*$ by $AZ_i^* = Y_i^*$, $i = 1, 2, 3$. From Example 4.3.14. in [1], we know that $\Omega := \{\tilde{r}\omega + \tilde{t}A\omega |\ \tilde{r} > 0,\ \tilde{t} \in \mathbf{R},\ \omega \in \mathfrak{g}_2^*\ with\ \|\omega\| = 1\}$ gives a cross-section for $\mathcal{O}_{Max}$. For each $\tilde{r}\omega + \tilde{t}A\omega \in \mathfrak{g}_{Max}^*$, there exists $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathcal{A}^* Z_3^* = \tilde{r}\omega + \tilde{t}A\omega$. Since $\mathbf{X}^* = \{(\rho_t^\sigma \circ \tau_r \circ (\sigma \otimes \sigma))^* |\ (t, r, \sigma) \in \mathbf{R} \times \mathbf{R}^+ \times \Sigma_0\}$ gives a cross-section for $\mathrm{Ad}_G^* \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(Z_3^*)$, there exists unique $(t, r, \sigma) \in \mathbf{R} \times \mathbf{R}^+ \times \Sigma_0$ such that $\mathcal{O}_{(\rho_t^\sigma \tau_r(\sigma\otimes\sigma))^* Z_3^*} = \mathcal{O}_{\mathcal{A}^* Z_3^*}$. So we have $\mathcal{O}_{(\rho_t^\sigma \tau_r(\sigma\otimes\sigma))^* Z_3^*} = \mathcal{O}_{\tilde{r}\omega + \tilde{t}A\omega}$, and hence there exists $X \in \mathfrak{g}$ such that

$$\begin{aligned}
\tilde{r}\omega + \tilde{t}A\omega &= \mathrm{Ad}_{\exp(X)}^* (\rho_t^\sigma \tau_r (\sigma \otimes \sigma))^* Z_3^* \\
&= \mathrm{Ad}_{\exp(X)}^* (\rho_t^\sigma \tau_r)^* (\sigma^* Z_3^*) \\
&= \mathrm{Ad}_{\exp(X)}^* \rho_t^{\sigma^*} (r\sigma^* Z_3^*) \\
&= \mathrm{Ad}_{\exp(X)}^* (r\sigma^* Z_3^* + tr\sigma^* Y_3^*) \\
&= (r\sigma^* Z_3^* + tr\sigma^* Y_3^*) + (\mathrm{ad}_X^* (r\sigma^* Z_3^* + tr\sigma^* Y_3^*) \\
&= r\sigma^* Z_3^* + tr\sigma^* Y_3^* + (\mathrm{ad}_X^* \circ (\sigma \otimes \sigma)^*)(rZ_3^* + trY_3^*) \\
&= r\sigma^* Z_3^* + tr\sigma^* Y_3^* + ((\sigma \otimes \sigma)^* \circ \mathrm{ad}_{(\sigma\otimes\sigma)^{-1}X}^*)(rZ_3^* + trY_3^*) \\
&= r\sigma^* Z_3^* + tr\sigma^* Y_3^* + (\sigma \otimes \sigma)^* \mathrm{ad}_{(\sigma\otimes\sigma)^{-1}X}^* (rZ_3^*) \\
&= r\sigma^* Z_3^* + tr\sigma^* Y_3^* + r(\sigma \otimes \sigma)^* \mathrm{ad}_{(\sigma\otimes\sigma)^{-1}X}^* Z_3^* \\
&= r\sigma^* Z_3^* + tr\sigma^* Y_3^* + r(\sigma \otimes \sigma)^* (c_1 Y_1^* + c_2 Y_2^*) \\
&= r\sigma^* Z_3^* + tr\sigma^* Y_3^* + rc_1\sigma^* Y_1^* + rc_2\sigma^* Y_2^*
\end{aligned}$$

for some constants $c_1, c_2 \in \mathbf{R}$. The seventh equality holds since we proved in Theorem 2.1 the identity $\mathcal{A}\mathrm{Ad}_x = \mathrm{Ad}_{\mathcal{A}x}\mathcal{A}$, $\forall x \in G$, $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$ (and $\mathrm{Aut}(G)$), it follows $\mathcal{A}\mathrm{ad}_X = \mathrm{ad}_{\mathcal{A}X}\mathcal{A}$. By replacing $\mathcal{A}$ by $\mathcal{A}^{-1}$, we get $\mathcal{A}\mathrm{ad}_{\mathcal{A}^{-1}X} = \mathrm{ad}_X\mathcal{A}$, and hence $\mathcal{A}^*\mathrm{ad}^*_{\mathcal{A}^{-1}X} = \mathrm{ad}^*_X\mathcal{A}^*$. Thus we have

$\tilde{r}\omega + \tilde{t}A\omega = r\sigma^* Z_3^* + tr\sigma^* Y_3^* + rc_1\sigma^* Y_1^* + rc_2\sigma^* Y_2^*$. it contains two conditions $\tilde{r}\omega = r\sigma^* Z_3^* \in \mathfrak{g}_2^*$ and $\tilde{t}A\omega = tr\sigma^* Y_3^* + rc_1\sigma^* Y_1^* + rc_2\sigma^* Y_2^* \in \mathfrak{g}_1^*$. The first condition follows $\omega = \sigma^* Z_3^*$ and $\underline{\tilde{r} = r}$. Since $\omega = \sigma^* Z_3^*$ and $A : Z_i^* \longmapsto Y_i^*$ is linear, we get $A\omega = \sigma^* Y_3^*$. So the second condition follows $\tilde{t}\sigma^* Y_3^* = tr\sigma^* Y_3^* + rc_1\sigma^* Y_1^* + rc_2\sigma^* Y_2^*$. Since $\{\sigma^* Y_1^*, \sigma^* Y_2^*, \sigma^* Y_3^*\}$ are linear independent, we get $\underline{\tilde{t} = tr}$ and $c_1 = c_2 = 0$. Hence there is a one-to-one correspondence between cross-section Ω of $\mathcal{O}_{Max}$ and $\mathbf{X}^*$ of $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(Z_3^*)$ with relations $\tilde{r} = r$ and $\tilde{t} = tr$.

From Example 4.3.14. in [1], we know $\tilde{r}^3 sin\varphi d\tilde{t}d\tilde{r}d\varphi d\theta$ is the Plancherel measure on $\mathcal{O}_{Max}$ viewed on cross-section Ω. Since $\mathfrak{g}_2^*$ is algebraically isomorphic to $\mathbf{R}^3$, give the rotation group on $\mathfrak{g}_2^*$ the measure $d\sigma$ which is area on the unit sphere. Thus $d\sigma = sin\varphi d\varphi d\theta$. This is indeed invariant under all rotations centered at the origin. Since $\tilde{t} = tr$, it follows $d\tilde{t} = rdt$, and hence

$$\begin{aligned} \tilde{r}^3 sin\varphi d\tilde{t}d\tilde{r}d\varphi d\theta &= r^4 sin\varphi dtdrd\varphi d\theta \\ &= r^4 dtdr(sin\varphi d\varphi d\theta) \\ &= r^4 dtdrd\sigma \end{aligned}$$

Then

$$d\mu(t, r, \sigma) := r^4 dtdrd\sigma$$

is the Plancherel measure on $\mathrm{Ad}_G^*\backslash\mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(Z_3^*)$ viewed on cross-section $\mathbf{X}^*$. Next we verify the relative invariance of μ.

For every $\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)$, $\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma) \in \mathbf{X}$, we know already

$$\mathrm{Ad}_G^*(\rho_{t_0}^{\sigma_0}\tau_{r_0}(\sigma_0 \otimes \sigma_0)\rho_t^{\sigma}\tau_r(\sigma \otimes \sigma))^* Z_3^* = \mathrm{Ad}_G^*(\rho^{\sigma_0\sigma}_{\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi}\tau_{r_0 r}(\sigma_0\sigma \otimes \sigma_0\sigma))^* Z_3^*$$

It follows that

$$\begin{aligned}
d\mu((t_0, r_0, \sigma_0)(t, r, \sigma)) &= d\mu((\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi, r_0 r, \sigma_0\sigma)) \\
&= (r_0 r)^4 d(\frac{t}{\sqrt{r_0}} + t_0 cos^2\psi) d(r_0 r) d(\sigma_0\sigma) \\
&= r_0^4 r^4 (\frac{1}{\sqrt{r_0}} dt)(r_0 dr) d\sigma \\
&= r_0^{\frac{9}{2}} (r^4 dt dr d\sigma) \\
&= |\det(\rho_{t_0}^{\sigma_0} \tau_{r_0} (\sigma_0 \otimes \sigma_0))^* | d\mu(t, r, \sigma)
\end{aligned}$$

Note that the equality $d(\sigma_0\sigma) = d\sigma$ holds because $d\sigma = sin\varphi d\varphi d\theta$ is rotation-invariant. Thus we know

$$\mu((\rho_{t_0}^{\sigma_0} \tau_{r_0} (\sigma_0 \otimes \sigma_0))^* E) = |\det(\rho_{t_0}^{\sigma_0} \tau_{r_0} (\sigma_0 \otimes \sigma_0))^* | \mu(E)$$

for every measurable subset E. Therefore Plancherel measure $\mu = r^4 dt dr d\sigma$ on the double coset space $\mathrm{Ad}_G^* \backslash \mathrm{Aut}^*(\mathfrak{g})/\mathrm{Stab}(Z_3^*)$ is relatively invariant for $\mathfrak{g} = \mathcal{F}_{3,2}$.

6. General Properties of Free 2-step Nilpotent Lie Algebras on n Generators

For any arbitrary nilpotent Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{g}^{(3)} = [\mathfrak{g}^{(2)}, \mathfrak{g}]$. If $\mathfrak{g}^{(3)} = \{0\}$, $\dim \mathfrak{g}^{(2)} = \frac{n(n-1)}{2}$, and $\dim \mathfrak{g} = n + \frac{n(n-1)}{2}$, then $\mathfrak{g}$ is algebraically isomorphic to $\mathcal{F}_{n,2}$, free nilpotent Lie algebra of two steps on n generators. The reason is the following:

Since $\dim(\mathfrak{g}/\mathfrak{g}^{(2)}) = \dim \mathfrak{g} - \dim \mathfrak{g}^{(2)} = n$, pick a basis $\{\bar{Y}_1, \bar{Y}_2, \cdots, \bar{Y}_n\}$ for $\mathfrak{g}/\mathfrak{g}^{(2)}$. Choose $Y_i \in \bar{Y}_i$, $\forall i = 1, \cdots, n$. Let

$\mathcal{B}_2 := \{Z_k |\ k = 1, 2, \cdots, \frac{n(n-1)}{2}\} = \{[Y_i, Y_j] |\ 1 \le i < j \le n\}$. Then $\mathcal{B}_2$ is a basis of $\mathfrak{g}^{(2)}$ since they span a space which is given as $\frac{n(n-1)}{2}$ dimension. And hence $\mathcal{B}_2 \cup \{Y_1, Y_2, \cdots, Y_n\}$ is a basis for $\mathfrak{g}$. Therefore $\mathfrak{g}$ is isomorphic to $\mathcal{F}_{n,2}$.

The above argument suggests all generators play the same role or they look similar with each other in some sense. So we expect their brackets look similar too. Let's explain this explicitly.

Let $\mathfrak{g} = \mathcal{F}_{n,2}$, and $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$. Give any arbitrary norm on $\mathfrak{g}$, let $\mathfrak{g}_1$ be the orthogonal complement of $\mathfrak{g}_2$. Pick a set of n orthonormal generators $\{Y_i | i = 1, \cdots, n\}$ in $\mathfrak{g}_1$. Then a question is raised: does the bracket preserve the norm and the orthogonality? In fact, it is not necessary. But we may adjust the norm in order to get a positive answer. The following lemma gives the answer in the affirmative. First, let's give a definition.

A norm is said to *respect* the Lie bracket if given any orthogonal map σ_1 of $\mathfrak{g}_1$ there exists an orthogonal map σ_2 of $\mathfrak{g}_2$ such that $\sigma_1 \otimes \sigma_2$ defines an automorphism of $\mathfrak{g}$.

Lemma 6.1. Let $\mathfrak{g} = \mathcal{F}_{n,2}$ $(n > 1)$, there exist norms on $\mathfrak{g}$ respecting the bracket. In other words, for any set of n orthonormal generators in the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$, all nontrivial brackets of any two generators are orthonormal.

Proof. Give any arbitrary norm $\|\cdot\|_1$ to $\mathfrak{g}$. Let $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$, and let $\mathfrak{g}_1$ be the orthogonal complement of $\mathfrak{g}_2$. Pick a set $\{Y_1, Y_2, \cdots, Y_n\}$ of orthonormal generators in $\mathfrak{g}_1$, and let $\{Z_1, Z_2, \cdots, Z_m\} = \{[Y_i, Y_j] | \ 1 \leq i < j \leq n\}$ where $m = \frac{n(n-1)}{2}$. Then $\mathfrak{g}_1 = Sp_{\mathbf{R}}\{Y_1, \cdots, Y_n\}$ and $\mathfrak{g}_2 = Sp_{\mathbf{R}}\{Z_1, \cdots, Z_m\}$. Keep the given norm $\|\cdot\|_1$ in $\mathfrak{g}_1$, and use Euclidean norm $\|\cdot\|_E$ in $\mathfrak{g}_2$ respecting $\{Z_1, \cdots, Z_m\}$. This means the corresponding inner product $\langle \ , \ \rangle_E$ defined by Euclidean norm $\|\cdot\|_E$ makes $\langle Z_i, Z_j \rangle_E = \delta_{ij}$, $\forall i, j = 1, 2, \cdots, n$. Then define the map $\|\cdot\| : \mathfrak{g} \longrightarrow \overline{\mathbf{R}^+}$ by

$$\|Y + Z\| = \sqrt{\|Y\|_1^2 + \|Z\|_E^2}$$

$\forall Y \in \mathfrak{g}_1$ and $Z \in \mathfrak{g}_2$. We claim that $\|\cdot\|$ is a norm on $\mathfrak{g}$.

(1) For every $W \in \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, let $W = Y + Z$, then $\|cW\| = \|c(Y + Z)\| = \|cY + cZ\| = \sqrt{\|cY\|_1^2 + \|cZ\|_E^2} = |c|\|Y + Z\| = |c|\|W\|$, $\forall c \in \mathbf{R}$.

$$\begin{aligned}
(2)\ \|W + W'\|^2 &= \|(Y + Z) + (Y' + Z')\|^2 \\
&= \|(Y + Y') + (Z + Z')\|^2 \\
&= \|Y + Y'\|_1^2 + \|Z + Z'\|_E^2 \\
&\leq (\|Y\|_1 + \|Y'\|_1)^2 + (\|Z\|_E + \|Z'\|_E)^2 \\
&= \|Y\|_1^2 + \|Y'\|_1^2 + \|Z\|_E^2 + \|Z'\|_E^2 + 2(\|Y\|_1\|Y'\|_1 + \|Z\|_E\|Z'\|_E) \\
&\leq \|Y\|_1^2 + \|Y'\|_1^2 + \|Z\|_E^2 + \|Z'\|_E^2 + 2\sqrt{\|Y\|_1^2 + \|Z\|_E^2}\sqrt{\|Y'\|_E^2 + \|Z'\|_E^2} \\
&= (\sqrt{\|Y\|_1^2 + \|Z\|_E^2} + \sqrt{\|Y'\|_E^2 + \|Z'\|_E^2})^2 \\
&= (\|W\| + \|W'\|)^2,
\end{aligned}$$

so we get $\|W + W'\| \leq \|W\| + \|W'\|$, $\forall W, W' \in \mathfrak{g}$.

(3) $\|W\| = 0 \Leftrightarrow \sqrt{\|Y\|_1^2 + \|Z\|_E^2} = 0 \Leftrightarrow Y = 0 = Z \Leftrightarrow W = 0$,

$\forall W = Y + Z \in \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

By (1), (2), and (3) we know $\|\cdot\| : \mathfrak{g} \longrightarrow \overline{\mathbf{R}^+}$ is a norm on $\mathfrak{g}$. Hence $\{Y_1, \cdots, Y_n, Z_1, \cdots Z_m\}$ is an orthonormal basis in $(\mathfrak{g}, \|\cdot\|)$ since $\{Y_1, \cdots, Y_n\}$ and $\{Z_1, \cdots Z_m\}$ are orthonormal bases in $(\mathfrak{g}_1, \|\cdot\|_1)$ and $(\mathfrak{g}_2, \|\cdot\|_E)$ respectively. Now choose any other orthonormal generators $Y_1', Y_2', \cdots, Y_n'$ in $(\mathfrak{g}_1, \|\cdot\|)$, let $\{Z_k' |\ k = 1, \cdots, m\} = \{[Y_i', Y_j'] |\ 1 \leq i < j \leq n\}$. We claim that $Z_1', Z_2', \cdots, Z_m'$ are orthonormal in $(\mathfrak{g}, \|\cdot\|)$.

It is sufficient to show that

(1) $[Y_1', Y_2']$ is perpendicular to $[Y_3', Y_4']$,

(2) $[Y_1', Y_2']$ is perpendicular to $[Y_2', Y_3']$,

(3) $[Y_1', Y_2']$ is a unit vector.

Since $\mathfrak{g}_1$ is spanned by $\{Y_1, \cdots, Y_n\}$, let

$$\begin{aligned}
Y_1' &= a_1Y_1 + a_2Y_2 + \cdots + a_nY_n \\
Y_2' &= b_1Y_1 + b_2Y_2 + \cdots + b_nY_n \\
Y_3' &= c_1Y_1 + c_2Y_2 + \cdots + c_nY_n \\
Y_4' &= d_1Y_1 + d_2Y_2 + \cdots + d_nY_n
\end{aligned}$$

, $\forall a_i, b_i, c_i, d_i \in \mathbf{R}$, $i = 1, 2, \cdots, n$. By assumption Y_1', Y_2', Y_3', Y_4' are orthonormal.

There are some helpful identities:

$\sum_{j=1}^n a_jb_j = 0$, $\sum_{j=1}^n a_jc_j = 0$, $\sum_{j=1}^n a_jd_j = 0$, $\sum_{j=1}^n b_jc_j = 0$, $\sum_{j=1}^n b_jd_j = 0$, $\sum_{j=1}^n a_j^2 = 1$, and $\sum_{j=1}^n b_j^2 = 1$.

(1) We first verify that the inner product of $[Y_1', Y_2']$ and $[Y_3', Y_4']$ is zero.

$\langle [Y_1', Y_2'], [Y_3', Y_4'] \rangle$

$= \langle \left[\sum_{i=1}^n a_iY_i, \sum_{j=1}^n b_jY_j\right], \left[\sum_{i=1}^n c_iY_i, \sum_{j=1}^n d_jY_j\right] \rangle$

$= \langle \sum_{i\neq j} a_i b_j [Y_i, Y_j], \sum_{i\neq j} c_i d_j [Y_i, Y_j]\rangle$

$= \langle \sum_{i<j}(a_i b_j - b_i a_j)[Y_i, Y_j], \sum_{i<j}(c_i d_j - d_i c_j)[Y_i, Y_j]\rangle$

$= \sum_{i<j}(a_i b_j - b_i a_j)(c_i d_j - d_i c_j)$

$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n}(a_i c_i b_j d_j + b_i d_i a_j c_j - b_i c_i a_j d_j - a_i d_i b_j c_j)$

$= \sum_{i=1}^{n-1} \left\{a_i c_i \sum_{j=i+1}^{n} b_j d_j + b_i d_i \sum_{j=i+1}^{n} a_j c_j - b_i c_i \sum_{j=i+1}^{n} a_j d_j - a_i d_i \sum_{j=i+1}^{n} b_j c_j\right\}$

$+a_n c_n \cdot 0 + b_n d_n \cdot 0 - b_n c_n \cdot 0 - a_n d_n \cdot 0$

$= \sum_{i=1}^{n-1} \{a_i c_i(-\sum_{j=1}^{i} b_j d_j) + b_i d_i(-\sum_{j=1}^{i} a_j c_j) - b_i c_i(-\sum_{j=1}^{i} a_j d_j) -$

$a_i d_i(-\sum_{j=1}^{i} b_j c_j)\}$

$+a_n c_n(-\sum_{j=1}^{n} b_j d_j) + b_n d_n(-\sum_{j=1}^{n} a_j c_j) - b_n c_n(-\sum_{j=1}^{n} a_j d_j) - a_n d_n(-\sum_{j=1}^{n} b_j c_j)$

$= \{a_1 c_1(-b_1 d_1) + b_1 d_1(-a_1 c_1) - b_1 c_1(-a_1 d_1) - a_1 d_1(-b_1 c_1)\}$

$+ \sum_{i=2}^{n} \{a_i c_i(-\sum_{j=1}^{i} b_j d_j) + b_i d_i(-\sum_{j=1}^{i} a_j c_j) - b_i c_i(-\sum_{j=1}^{i} a_j d_j) -$

$a_i d_i(-\sum_{j=1}^{i} b_j c_j)\}$

$= \{0\} + \sum_{i=2}^{n} \{a_i c_i(-\sum_{j=1}^{i-1} b_j d_j) + b_i d_i(-\sum_{j=1}^{i-1} a_j c_j) - b_i c_i(-\sum_{j=1}^{i-1} a_j d_j) -$

$a_i d_i(-\sum_{j=1}^{i-1} b_j c_j)\}$

$+ \sum_{i=2}^{n} \{a_i c_i(-b_i d_i) + b_i d_i(-a_i c_i) - b_i c_i(-a_i d_i) - a_i d_i(-b_i c_i)\}$

$= -\sum_{i=2}^{n} \sum_{j=1}^{i-1} \{a_i c_i b_j d_j + b_i d_i a_j c_j - b_i c_i a_j d_j - a_i d_i b_j c_j\}$

$= -\sum_{j=1}^{n-1} \sum_{i=j+1}^{n} \{a_i c_i b_j d_j + b_i d_i a_j c_j - b_i c_i a_j d_j - a_i d_i b_j c_j\}$

$= -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \{a_j c_j b_i d_i + b_j d_j a_i c_i - b_j c_j a_i d_i - a_j d_j b_i c_i\}$

$= -\sum_{i<j}(a_i b_j - b_i a_j)(c_i d_j - d_i c_j)$

$= -\langle [Y_1', Y_2'], [Y_3', Y_4']\rangle$

Thus, we get $\langle [Y_1', Y_2'], [Y_3', Y_4']\rangle = -\langle [Y_1', Y_2'], [Y_3', Y_4']\rangle$, and hence $\langle [Y_1', Y_2'], [Y_3', Y_4']\rangle = 0$.

(2) We now verify that the inner product of $[Y_1', Y_2']$ and $[Y_2', Y_3']$ is zero. Actually we replace Y_4' by $-Y_2'$ in (1), in other words, replace d_j by $-b_j$ in (1), then we get

$$\langle [Y_1', Y_2'], [Y_2', Y_3']\rangle = \langle [Y_1', Y_2'], [Y_3', -Y_2']\rangle = -\sum_{i<j}(a_i b_j - b_i a_j)(c_i(-b_j) - (-b_i)c_j)$$
$$= -\sum_{i<j}(a_i b_j - b_i a_j)(b_i c_j - c_i b_j) = -\langle [Y_1', Y_2'], [Y_2', Y_3']\rangle$$

So we have $\langle [Y_1', Y_2'], [Y_2', Y_3'] \rangle = -\langle [Y_1', Y_2'], [Y_2', Y_3'] \rangle$, and hence $\langle [Y_1', Y_2'], [Y_2', Y_3'] \rangle = 0$.

(3) We now verify that $\|[Y_1', Y_2']\|^2 = 1$.

$$
\begin{aligned}
&\|[Y_1', Y_2']\|^2 \\
&= \|[\textstyle\sum_{i=1}^n a_i Y_i, \sum b_j Y_j]\|^2 \\
&= \left\| \textstyle\sum_{i<j} (a_i b_j - b_i a_j)[Y_i, Y_j] \right\|^2 \\
&= \textstyle\sum_{i<j} (a_i b_j - b_i a_j)^2 \\
&= \textstyle\sum_{i=1}^{n-1} \sum_{j=i+1}^n (a_i^2 b_j^2 + b_i^2 a_j^2 - 2a_i b_i a_j b_j) \\
&= \textstyle\sum_{i=1}^{n-1} \left\{ a_i^2 \sum_{j=i+1}^n b_j^2 + b_i^2 \sum_{j=i+1}^n a_j^2 - 2a_i b_i \sum_{j=i+1}^n a_j b_j \right\} + \\
&\{a_n^2 \cdot 0 + b_n^2 \cdot 0 - 2a_n b_n \cdot 0\} \\
&= \textstyle\sum_{i=1}^{n-1} \left\{ a_i^2 (1 - \sum_{j=1}^i b_j^2) + b_i^2 (1 - \sum_{j=1}^i a_j^2) - 2a_i b_i (-\sum_{j=1}^i a_j b_j) \right\} \\
&\quad + \left\{ a_n^2 \left(1 - \textstyle\sum_{j=1}^n b_j^2\right) + b_n^2 \left(1 - \sum_{j=1}^n a_j^2\right) - 2a_n b_n \left(-\sum_{j=1}^n a_j b_j\right) \right\} \\
&= \{a_1^2 (1 - b_1^2) + b_1^2 (1 - a_1^2) - 2a_1 b_1 (-a_1 b_1)\} \\
&\quad + \textstyle\sum_{i=2}^n \left\{ a_i^2 (1 - \sum_{j=1}^i b_j^2) + b_i^2 (1 - \sum_{j=1}^i a_j^2) - 2a_i b_i (-\sum_{j=1}^i a_j b_j) \right\} \\
&= \textstyle\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 - \sum_{i=2}^n \left\{ a_i^2 \sum_{j=1}^i b_j^2 + b_i^2 \sum_{j=1}^i a_j^2 - 2a_i b_i \sum_{j=1}^i a_j b_j \right\} \\
&= 1 + 1 - \textstyle\sum_{i=2}^n \sum_{j=1}^i \{a_i^2 b_j^2 + b_i^2 a_j^2 - 2a_i b_i a_j b_j\} \\
&= 2 - \textstyle\sum_{i=2}^n \sum_{j=1}^{i-1} \{a_i^2 b_j^2 + b_i^2 a_j^2 - 2a_i b_i a_j b_j\} \\
&= 2 - \textstyle\sum_{j=1}^{n-1} \sum_{i=j+1}^n \{a_i^2 b_j^2 + b_i^2 a_j^2 - 2a_i b_i a_j b_j\} \\
&= 2 - \textstyle\sum_{i=1}^{n-1} \sum_{j=i+1}^n \{a_j^2 b_i^2 + b_j^2 a_i^2 - 2a_j b_j a_i b_i\} \\
&= 2 - \textstyle\sum_{i<j} (a_i b_j - b_i a_j)^2 \\
&= 2 - \|[Y_1', Y_2']\|^2
\end{aligned}
$$

Thus we get $\|[Y_1', Y_2']\|^2 = 2 - \|[Y_1', Y_2']\|^2$, and hence $\|[Y_1', Y_2']\|^2 = 1$.

We verified (1), (2), and (3), then this proved the claim. Therefore for any orthonormal generators $\{Y_1', Y_2', \cdots, Y_n'\}$, their brackets $\{[Y_i', Y_j'] : 1 \le i < j \le n\}$ are also orthonormal with respect to the norm $\|\cdot\|$.

Motivation. Let $\mathfrak{g} = \mathcal{F}_{n,2}$. The previous lemma guarantees that there always exist norms on $\mathfrak{g}$ respecting the bracket. So now we give a norm on $\mathfrak{g}$ respecting the bracket. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$, and use the operator norm on $\mathfrak{g}^*$. Choose an orthonormal basis $\mathcal{B}$ for $\mathfrak{g}$, let $\mathcal{B}^*$ be its dual basis in $\mathfrak{g}^*$. Then the question is that whether $\mathcal{B}^*$ is orthonormal or not? The following lemma answers this question saying 'yes'. And more than that is the orthonormality of both bases are equivalent for any finite-dimensional vector space $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$.

Lemma 6.2. If $\mathfrak{g}$ is an n-dimensional real vector space, let $\mathfrak{g}^*$ be its dual space. Give any arbitrary norm on $\mathfrak{g}$, and use the corresponding operator norm on $\mathfrak{g}^*$. If $\mathcal{B} = \{X_1, X_2, \cdots, X_n\}$ is a basis of $\mathfrak{g}$, let $\mathcal{B}^* = \{X_1^*, X_2^*, \cdots, X_n^*\}$ be its dual basis. Then the orthonormality of $\mathcal{B}$ and $\mathcal{B}^*$ in $\mathfrak{g}$ and $\mathfrak{g}^*$ are equivalent.

Proof. We first claim that if $\mathcal{B}$ is an orthonormal basis of $\mathfrak{g}$, then so is $\mathcal{B}^*$ for $\mathfrak{g}^*$. Let $\|\cdot\|$ be any given norm on $\mathfrak{g}$, and let $\|\cdot\|_{op}$ be the corresponding operator norm on $\mathfrak{g}^*$. Since $\mathcal{B} = \{X_1, \cdots, X_n\}$ is a basis of $\mathfrak{g}$, for each $X \in \mathfrak{g}$, let $X = \sum_{j=1}^n \alpha_j X_j$. Since $\mathcal{B}$ is orthonormal, it follows $\|X\| = \sqrt{\sum_{j=1}^n \alpha_j^2}$. Since $\mathcal{B}^*$ is the dual basis of $\mathcal{B}$, it means $X_i^*(X_j) = \delta_{ij}$, $\forall i, j = 1, 2, \cdots n$. So $X_i^*(X) = X_i^*(\sum_{j=1}^n \alpha_j X_j) = \sum_{j=1}^n \alpha_j X_i^*(X_j) = \alpha_i$, $\forall i = 1, 2, \cdots, n$.

Then

$\|X_i^*\|_{op} = \sup_{\|X\|=1} |X_i^*(X)| = \sup_{\sqrt{\sum_{j=1}^n \alpha_j^2}=1} |\alpha_i| = 1$, $\forall i = 1, 2, \cdots, n$.

Thus, the orthonormality of $\mathcal{B}$ implies that the unit length of every basis element of $\mathcal{B}^*$. We now want to verify the orthogonality of $\mathcal{B}^*$. It suffices to show that $\|X_i^* + X_j^*\|_{op} = \|X_i^* - X_j^*\|_{op}$, $\forall 1 \leq i \neq j \leq n$. If this is done, it means that the inner product of any distinct basis elements is zero. In other words, all basis elements of $\mathcal{B}^*$ are mutually perpendicular.

For every $X \in \mathfrak{g}$, let $X = \sum_{k=1}^{n} \alpha_k X_k$. Then we have

$\|X_i^* + X_j^*\|_{op} = \sup_{\|X\|=1} |(X_i^* + X_j^*)(X)| = \sup_{\sqrt{\sum_{k=1}^{n} \alpha_k^2}=1} |\alpha_i + \alpha_j| = \sqrt{2}$

when $\alpha_i = \alpha_j = \overset{+}{-} \frac{1}{\sqrt{2}}$ and other α_k's are zero. Meanwhile, we also have

$\|X_i^* - X_j^*\|_{op} = \sup_{\|X\|=1} |(X_i^* - X_j^*)(X)| = \sup_{\sqrt{\sum_{k=1}^{n} \alpha_k^2}=1} |\alpha_i - \alpha_j| = \sqrt{2}$

when $\alpha_i = -\alpha_j = \overset{+}{-} \frac{1}{\sqrt{2}}$ and other α_k's are zero. So we get

$\|X_i^* + X_j^*\|_{op} = \|X_i^* - X_j^*\|_{op}$ as desired, and hence we know the orthonormality of $\mathcal{B}$ in $(\mathfrak{g}, \|\cdot\|)$ implies the orthonormality of $\mathcal{B}^*$ in $(\mathfrak{g}^*, \|\cdot\|_{op})$. Next we claim that if $\mathcal{B}^*$ is an orthonormal basis of $\mathfrak{g}^*$, then so is $\mathcal{B}$ in $\mathfrak{g}$.

Let $\mathfrak{g}^{**}$ be the dual space of $\mathfrak{g}^*$. For every $X \in \mathfrak{g}$, define $X^{**}(l) = l(X)$, $\forall l \in \mathfrak{g}^*$. Then the map $X \longmapsto X^{**}$ is an isomorphism from $\mathfrak{g}$ onto $\mathfrak{g}^{**}$ since $\mathfrak{g}$ is finite-dimensional. Hence we can identify X^{**} with X, $\forall X^{**} \in \mathfrak{g}^{**}$. Let $\mathcal{B}^{**} = \{X_1^{**}, X_2^{**}, \cdots, X_n^{**}\}$ be the dual basis of $\mathcal{B}^* = \{X_1^*, X_2^*, \cdots, X_n^*\}$. This means $X_i^{**}(X_j^*) = \delta_{ij}$, i.e., $X_j^*(X_i) = \delta_{ij}$, $\forall 1 \le i, j \le n$. We denote the operator norm on $\mathfrak{g}^{**}$ by $\|\cdot\|_{\sup}$, so

$$\|X^{**}\|_{\sup} = \sup_{\|l\|_{op}=1} |X^{**}(l)| = \sup_{\|l\|_{op}=1} |l(X)|$$

By applying the argument of the first claim we know that the orthonormality of $\mathcal{B}^*$ in $(\mathfrak{g}^*, \|\cdot\|_{op})$ implies the orthonormality of $\mathcal{B}^{**}$ in $(\mathfrak{g}^{**}, \|\cdot\|_{\sup})$. Since every $X^{**} \in \mathfrak{g}^{**}$ can be identified with $X \in \mathfrak{g}$, it follows that the orthonormality of $\mathcal{B}^*$ in $(\mathfrak{g}^*, \|\cdot\|_{op})$ implies the orthonormality of $\mathcal{B}$ in $(\mathfrak{g}, \|\cdot\|_{\sup})$. So now the question is that whether the operator norm $\|\cdot\|_{\sup}$ on $\mathfrak{g}$ obtained by identifying $\mathfrak{g}$ with $\mathfrak{g}^{**}$ is the same as the original given norm $\|\cdot\|$ on $\mathfrak{g}$ or not? The answer is positive. Since $\mathcal{B} = \{X_1, \cdots, X_n\}$ is an orthonormal basis of $(\mathfrak{g}, \|\cdot\|)$, for every $X \in \mathfrak{g}$, let $X = \sum_{j=1}^{n} \alpha_j X_j$, then $\|X\| = \|\sum_{j=1}^{n} \alpha_j X_j\| = \sqrt{\sum_{j=1}^{n} \alpha_j^2}$. Meanwhile, we also know $\mathcal{B}$ is an orthonormal basis of $(\mathfrak{g}, \|\cdot\|_{\sup})$ by the identification of $\mathfrak{g}$ with $\mathfrak{g}^{**}$.

So $\|X\|_{\sup} = \left\|\sum_{j=1}^{n}\alpha_j X_j\right\|_{\sup} = \sqrt{\sum_{j=1}^{n}\alpha_j^2}$. Thus, we get $\|X\| = \|X\|_{\sup}$, $\forall X \in \mathfrak{g}$. Hence the operator norm on $\mathfrak{g} \cong \mathfrak{g}^{**}$ is the same as arbitrary given norm on $\mathfrak{g}$. By both claims we conclude that the orthonormality of $\mathcal{B}$ in $(\mathfrak{g}, \|\cdot\|)$ and $\mathcal{B}^*$ in $(\mathfrak{g}^*, \|\cdot\|_{op})$ are equivalent for any arbitrary norm $\|\cdot\|$ on $\mathfrak{g}$ and the corresponding operator norm $\|\cdot\|_{op}$ on $\mathfrak{g}^*$.

Motivation. For $\mathfrak{g} = \mathcal{F}_{n,2}$, the dimension of its center is $m = \frac{n(n-1)}{2}$. Let $Y_1, Y_2, \cdots, Y_n$ be n generators of $\mathfrak{g}$, and let
$\{Z_1, Z_2, \cdots, Z_m\} := \{[Y_i, Y_j] \mid 1 \le i < j \le n\}$, then $\mathcal{B} := \{Z_1, \cdots, Z_m, Y_1, \cdots, Y_n\}$ is a basis of $\mathfrak{g}$.
For every $\mathcal{A} \in \mathrm{Aut}(\mathfrak{g})$, $\mathcal{A}$ has the $(m+n) \times (m+n)$ matrix relative to the basis $\mathcal{B}$

$$[\mathcal{A}]_{\mathcal{B}} := \begin{bmatrix} Q_{m\times m} & R_{m\times n} \\ 0_{n\times m} & P_{n\times n} \end{bmatrix}$$

and each entry of $Q_{m\times m}$ is a polynomial in the variables of entries of $P_{n\times n}$. We next show the relation between determinants of $P_{n\times n}$ and $Q_{m\times m}$.

Lemma 6.3. For $\mathfrak{g} = \mathcal{F}_{n,2}$, let $Y_1, Y_2, \cdots, Y_n$ be n generators of $\mathfrak{g}$, and let the space $\mathcal{S}$ be spanned by all generators Y_i. For each invertible linear operator P on $\mathcal{S}$, define the map $Q : [\mathfrak{g}, \mathfrak{g}] \longrightarrow [\mathfrak{g}, \mathfrak{g}]$ by $Q([Y_i, Y_j]) = [PY_i, PY_j]$, $\forall 1 \le i < j \le n$, then we have

$$\det Q = (\det P)^{n-1}$$

Proof. Let $\mathcal{B}_2 := \{[Y_i, Y_j] \mid 1 \le i < j \le n\}$, then $\mathcal{B}_2$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. There are two cases depending upon the positive integer n.

Case 1. Suppose n is even.

For each invertible linear operator P on $\mathcal{S}$, let detQ be viewed as a function of n column vectors $PY_1, PY_2, \cdots, PY_n$. Define a function $F : \mathcal{S}^n \longrightarrow \mathbf{R}$ by

$$F(PY_1, PY_2, \cdots, PY_n) = (\det Q)^{\frac{1}{n-1}}$$

then claim that $F = \det P$. If this is done, then the result $\det Q = (\det P)^{n-1}$ follows immediately. Actually it is equivalent to show that F is an alternating n-linear form on $\mathcal{S}$ which maps $(Y_1, \cdots, Y_n)$ to 1.

(1) If P is the identity operator on $\mathcal{S}$, by definition $Q([Y_i, Y_j]) = [PY_i, PY_j] = [Y_i, Y_j]$. Then Q is the identity operator on $[\mathfrak{g}, \mathfrak{g}]$, and hence $F(Y_1, \cdots, Y_n) = (\det I_{m\times m})^{\frac{1}{n-1}} = 1$.

(2) Let $P' : \mathcal{S} \longrightarrow \mathcal{S}$ be the operator such that $P'Y_k = cPY_k$ for some non-zero constant c, and $P'Y_i = PY_i$, $\forall i \neq k$. And let Q' be the corresponding operator of P' by definition. Since $P'Y_k$ appears in brackets with exactly n-1 other vectors $P'Y_i$, it follows that $\det Q' = c^{n-1} \det Q$, and hence

$$F(PY_1, \cdots, PY_{k-1}, cPY_k, PY_{k+1}, \cdots, PY_n) = (c^{n-1}\det Q)^{\frac{1}{n-1}} = c(\det Q)^{\frac{1}{n-1}} = cF(PY_1, \cdots, PY_n)$$

(3) Let $P' : \mathcal{S} \longrightarrow \mathcal{S}$ be the operator such that $P'Y_i = PY_j$ for some $i \neq j$, and $P'Y_s = PY_s$ for all $s \neq i$. Let Q' be the corresponding operator of P', then $Q'([Y_i, Y_j]) = [P'Y_i, P'Y_j] = [PY_j, PY_j] = 0$. So one column of matrix $[Q']_{\mathcal{B}_2}$ is zero vector, and hence $\det Q' = 0$. It follows

$$F(PY_1, \cdots, \underbrace{PY_j}_{\text{the } i \text{ th term}}, \cdots PY_j, \cdots, PY_n) =$$

$$F(P'Y_1, \cdots, P'Y_i, \cdots P'Y_j, \cdots P'Y_n) = (\det Q')^{\frac{1}{n-1}} = 0$$

(4) In order to show linearity of F in each variable, we verify

$$F(PY_1, \cdots, PY_{i-1}, Y + cY', PY_{i+1}, \cdots, PY_n) =$$

$F(PY_1,\cdots,PY_{i-1},Y,PY_{i+1},\cdots,PY_n)+cF(PY_1,\cdots PY_{i-1},Y',PY_{i+1},\cdots,PY_n)$ for every $Y,Y'\in\mathcal{S}$ and $c\in\mathbf{R}$ where $Y+cY'$, Y, and Y' are in the i <u>th</u> component. Since P is bijective on the space $\mathcal{S}=\mathbf{R}-span\{Y_1,\cdots,Y_n\}$, every vector in $\mathcal{S}$ can be written as $\sum_{j=1}^n c_jPY_j$. So it is sufficient to show property (2) and $F(PY_1,\cdots,\underbrace{PY_i+PY_j}_{\text{the } i \text{ th term}},\cdots,PY_n)=$
$F(PY_1,\cdots,PY_i,\cdots,PY_n)+F(PY_1,\cdots,\underbrace{PY_j}_{\text{the } i \text{ th term}},\cdots,PY_n)$. Since we already finished the proof of property (2), let's verify the above identity.

Let $P':\mathcal{S}\longrightarrow\mathcal{S}$ be the operator such that $P'Y_i=PY_i+PY_j$ for some $i\neq j$, and $P'Y_s=PY_s$ for all $s\neq i$. And let Q' be the corresponding operator of P'. Then by definition $F(PY_1,\cdots,PY_{i-1},PY_i+PY_j,PY_{i+1},\cdots,PY_n)=F(P'Y_1,\cdots,P'Y_n)=(\det Q')^{\frac{1}{n-1}}$ Since the difference between $[Q']_{\mathcal{B}_2}$ and $[Q]_{\mathcal{B}_2}$ are the columns of the brackets of PY_i+PY_j with other PY_k's for $k\neq i$, we next figure out what $\det Q'$ is by considering each bracket of PY_i+PY_j with PY_k for $k\neq i$.

For $k=1$ the bracket $[PY_1,PY_i+PY_j]=[PY_1,PY_i]+[PY_1,PY_j]$, it follows that $\det Q'=\det Q'_1+\det Q'_2$ where $[Q'_1]_{\mathcal{B}_2}$ and $[Q'_2]_{\mathcal{B}_2}$ are obtained by replacing the column $Q'([PY_1,PY_i+PY_j])$ of $[Q']_{\mathcal{B}_2}$ by $Q'([PY_1,PY_i])$ and $Q'([PY_1,PY_j])$ respectively. Notice that the vector PY_i+PY_j is on the i <u>th</u> column of matrix $[P']_{\mathcal{B}_1}$ where basis $\mathcal{B}_1=\{Y_1,\cdots,Y_n\}$. So $[Q'_2]_{\mathcal{B}_2}$ has two identical columns $Q'_2([PY_1,PY_j])$, and hence $\det Q'_2=0$. Thus we get $\det Q'=\det Q'_1$, and note that $[Q'_1]_{\mathcal{B}_2}$ has the same columns as $[Q]_{\mathcal{B}_2}$ except the columns $Q'_1([PY_i+PY_j,PY_s])$ for $s=2,\cdots,i-1,i+1,\cdots,n$.

For $k=2$ the bracket $[PY_2,PY_i+PY_j]=[PY_2,PY_i]+[PY_2,PY_j]$, it follows that $\det Q'_1=\det Q''_1+\det Q''_2$ where Q''_1 and Q''_2 are obtained by replacing the column $Q'_1([PY_2,PY_i+PY_j])$ of $[Q'_1]_{\mathcal{B}_2}$ by $Q'_1([PY_2,PY_i])$ and $Q'_1([PY_2,PY_j])$ respectively. Then $[Q''_2]_{\mathcal{B}_2}$ has two identical columns $Q''_2([PY_2,PY_j])$, and hence

$\det Q_2'' = 0$. Thus we get $\det Q' = \det Q_1' = \det Q_1''$, and note that $[Q_1'']_{\mathcal{B}_2}$ has the same columns as $[Q]_{\mathcal{B}_2}$ except the columns $Q_1''([PY_i + PY_j, PY_s])$ for $s = 3, \cdots, i-1, i+1, \cdots, n$.

Similar procedure applies to $k = 3, \cdots, i-1, i+1, \cdots, n$, then we get $\det Q' = \det Q_1' = \det Q_1'' = \cdots = \det Q_1^{(n-1)}$, and $[Q_1^{(n-1)}]_{\mathcal{B}_2}$ has the same columns as $[Q]_{\mathcal{B}_2}$. Hence $\det Q' = \det Q$. Thus

$F(PY_1, \cdots, PY_{i-1}, PY_i + PY_j, PY_{i+1}, \cdots, PY_n) = (\det Q')^{\frac{1}{n-1}} = (\det Q)^{\frac{1}{n-1}}$

$= F(PY_1, \cdots, PY_i, \cdots, PY_j, \cdots, PY_n)$

$= F(PY_1, \cdots, PY_i, \cdots, PY_j, \cdots, PY_n) + F(PY_1, \cdots, PY_j, \cdots, PY_j, \cdots, PY_n)$

since $F(PY_1, \cdots, PY_j, \cdots, PY_j, \cdots, PY_n) = 0$ by property (3). Therefore F is an n-linear function.

(5) We now verify F is alternating.

$$0 = F(PY_1, \cdots, \underbrace{PY_i + PY_j}_{the\ i\ \underline{th}\ term}, \cdots, \underbrace{PY_i + PY_j}_{the\ j\ \underline{th}\ term}, \cdots, PY_n)$$

$$= F(PY_1, \cdots, PY_i, \cdots, PY_i, \cdots PY_n) + F(PY_1, \cdots, PY_i, \cdots, PY_j, \cdots PY_n)$$

$$+F(PY_1, \cdots, PY_j, \cdots, PY_i, \cdots PY_n) + F(PY_1, \cdots, PY_j, \cdots, PY_j, \cdots PY_n)$$

$$= F(PY_1, \cdots, PY_i, \cdots, PY_j, \cdots PY_n) + F(PY_1, \cdots, PY_j, \cdots, PY_i, \cdots PY_n)$$

It follows that

$F(PY_1, \cdots, PY_i, \cdots, PY_j, \cdots PY_n) = -F(PY_1, \cdots, PY_j, \cdots, PY_i, \cdots PY_n)$

Hence F is alternating. Thus we know F is an alternating n-linear function which maps $(Y_1, \cdots, Y_n)$ to 1, so $F(PY_1, \cdots, PY_n) = \det P$. Therefore $\det Q = (\det P)^{n-1}$.

Case 2. Suppose n is odd.

For each invertible linear operator P on $\mathcal{S}$, define a function $F : \mathcal{S}^n \longrightarrow \mathbf{R}$ by $F(PY_1, \cdots, PY_n) = (\det P \det Q)^{\frac{1}{n}}$ where Q is the corresponding operator of P

defined as before. We claim that $F = \det P$. If this is done, then we get $(\det P \det Q)^{\frac{1}{n}} = \det P$, and hence $\det P \det Q = (\det P)^n$. So $\det Q = (\det P)^{n-1}$.

(1) If P is the identity operator on $\mathcal{S}$, so is Q. Then
$F(Y_1, \cdots, Y_n) = (\det I_{n\times n} \det I_{\frac{n(n-1)}{2}\times\frac{n(n-1)}{2}})^{\frac{1}{n}} = 1$.

(2) The same assumption as in the property (2) of the first case, if $P'Y_k = cPY_k$ for some index k, we obtain

$F(PY_1, \cdots PY_{k-1}, cPY_k, PY_{k+1}, \cdots PY_n) = F(P'Y_1, \cdots, P'Y_n) = (\det P' \det Q')^{\frac{1}{n}}$
$= ((c \det P)(c^{n-1} \det Q))^{\frac{1}{n}} = c(\det P \det Q)^{\frac{1}{n}} = cF(PY_1, \cdots, PY_k, \cdots, PY_n)$

(3) Let $P' : \mathcal{S} \longrightarrow \mathcal{S}$ be the operator such that $P'Y_i = PY_j$ for some $i \neq j$, and $P'Y_s = PY_s$ for all $s \neq i$. Then $\det P' = 0$ since matrix $[P']_{\mathcal{B}_1}$ has two identical column PY_j where $\mathcal{B}_1 = \{Y_1, \cdots, Y_n\}$. Let Q' be the corresponding operator of P'. Then

$F(PY_1, \cdots, \underbrace{PY_j}_{the\ i\ \underline{th}\ term}, \cdots, \underbrace{PY_j}_{the\ j\ \underline{th}\ term}, \cdots PY_n) = F(P'Y_1, \cdots, P'Y_n) =$
$(\det P' \det Q')^{\frac{1}{n}} = 0$

(4) In order to show the linearity of F, it suffices to verify property (2) and

$F(PY_1, \cdots, \underbrace{PY_i + PY_j}_{the\ i\ \underline{th}\ term}, \cdots, PY_n)$
$= F(PY_1, \cdots, PY_i, \cdots, PY_n) + F(PY_1, \cdots, \underbrace{PY_j}_{the\ i\ \underline{th}\ term}, \cdots, PY_n)$

Let $P' : \mathcal{S} \longrightarrow \mathcal{S}$ be the operator such that $P'Y_i = PY_i + PY_j$ for some $i \neq j$, and $P'Y_s = PY_s$ for all $s \neq i$. Then $\det P' = \det P + \det P_0$ where $[P_0]_{\mathcal{B}_1}$ is the matrix after replacing the i $\underline{th}$ column PY_i of $[P]_{\mathcal{B}_1}$ by PY_j. So $\det P_0 = 0$, and hence $\det P' = \det P$. Let Q' be the corresponding operator of P'. The same reason as property (4) of the first case leads to the fact $\det Q' = \det Q$. Hence

$F(PY_1, \cdots, \underbrace{PY_i + PY_j}_{the\ i\ \underline{th}\ term}, \cdots, PY_n) = F(P'Y_1, \cdots, P'Y_i, \cdots, P'Y_n)$

$= (\det P' \det Q')^{\frac{1}{n}} = (\det P \det Q)^{\frac{1}{n}} = F(PY_1, \cdots, PY_i, \cdots, PY_n)$

$= F(PY_1, \cdots, PY_i, \cdots, PY_n) + F(PY_1, \cdots, \underbrace{PY_j}_{\textit{the i th term}}, \cdots, PY_n)$

Therefore F is an n-linear function.

(5) The same argument as in the property (5) of the first case leads to the fact that F is alternating. Hence F is an alternating n-linear function which maps $(Y_1, \cdots, Y_n)$ to 1. So $F(PY_1, \cdots, PY_n) = \det P$, i.e., $(\det P \det Q)^{\frac{1}{n}} = \det P$. Therefore $\det Q = (\det P)^{n-1}$.

Lemma 6.4. For each $n \times n$ matrix R and almost all $n \times n$ matrix P, there exists a unique $n \times n$ skew-symmetric matrix S such that $R + SP$ is an upper-triangular matrix.

Proof. Given $R = (r_{ij})_{n \times n}$, $P = (p_{ij})_{n \times n}$, let

$$S = \begin{bmatrix} 0 & -s_{2,1} & -s_{3,1} & \cdots & -s_{n,1} \\ s_{2,1} & 0 & -s_{3,2} & \cdots & -s_{n,2} \\ s_{3,1} & s_{3,2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -s_{n,n-1} \\ s_{n,1} & s_{n,2} & \cdots & s_{n,n-1} & 0 \end{bmatrix}$$

Consider the last row of the $n \times n$ matrix $SP + R$, let $(SP + R)_{n,i} = 0$, $\forall i = 1, 2, \cdots, n-1$.

Then we have n-1 linear equations with n-1 variables $s_{n,1}, s_{n,2}, \cdots, s_{n,n-1}$:

$$\begin{aligned} s_{n,1}p_{1,1} + s_{n,2}p_{2,1} + \cdots + s_{n,n-1}p_{n-1,1} + 0 \cdot p_{n,1} &= -r_{n,1} \\ s_{n,1}p_{1,2} + s_{n,2}p_{2,2} + \cdots + s_{n,n-1}p_{n-1,2} + 0 \cdot p_{n,2} &= -r_{n,2} \\ \vdots \quad &\vdots \quad \vdots \\ s_{n,1}p_{1,n-1} + s_{n,2}p_{2,n-1} + \cdots + s_{n,n-1}p_{n-1,n-1} + 0 \cdot p_{n,n-1} &= -r_{n,n-1} \end{aligned}$$

Let

$$P_{n-1} := \begin{bmatrix} p_{1,1} & p_{2,1} & \cdots & p_{n-1,1} \\ p_{1,2} & p_{2,2} & \cdots & p_{n-1,2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,n-1} & p_{2,n-1} & \cdots & p_{n-1,n-1} \end{bmatrix}$$

If $\det P_{n-1} \neq 0$, then there exist unique solutions $(s_{n,1}, s_{n,2}, \cdots, s_{n,n-1})$ with $s_{n,i} = \frac{\det P^i_{n-1}}{\det P_{n-1}}$ $\forall i = 1, 2, \cdots, n-1$, where each P^i_{n-1} is the $(n-1) \times (n-1)$ matrix of replacing the i <u>th</u> column of P_{n-1} by $\begin{pmatrix} -r_{n,1} \\ -r_{n,2} \\ \vdots \\ -r_{n,n-1} \end{pmatrix}$

Consider the n-1 <u>th</u> row of $n \times n$ matrix $SP + R$, let $(SP + R)_{n-1,i} = 0$, $\forall i = 1, 2, \cdots, n-2$. Then we have n-2 linear equations with n-2 variables $s_{n-1,1}, s_{n-1,2}, \cdots, s_{n-1,n-2}$:

$$\begin{aligned}
s_{n-1,1}p_{1,1} + \cdots + s_{n-1,n-2}p_{n-2,1} + 0 \cdot p_{n-1,1} - s_{n,n-1}p_{n,1} + r_{n-1,1} &= 0 \\
s_{n-1,1}p_{1,2} + \cdots + s_{n-1,n-2}p_{n-2,2} + 0 \cdot p_{n-1,2} - s_{n,n-1}p_{n,2} + r_{n-1,2} &= 0 \\
\vdots \quad &\vdots \quad \vdots \\
s_{n-1,1}p_{1,n-2} + \cdots + s_{n-1,n-2}p_{n-2,n-2} + 0 \cdot p_{n-1,n-2} - s_{n,n-1}p_{n,n-2} + r_{n-1,n-2} &= 0
\end{aligned}$$

It follows that

$$\begin{aligned}
s_{n-1,1}p_{1,1} + \cdots + s_{n-1,n-2}p_{n-2,1} &= s_{n,n-1}p_{n,1} - r_{n-1,1} \\
s_{n-1,1}p_{1,2} + \cdots + s_{n-1,n-2}p_{n-2,2} &= s_{n,n-1}p_{n,2} - r_{n-1,2} \\
\vdots \quad &\vdots \quad \vdots \\
s_{n-1,1}p_{1,n-2} + \cdots + s_{n-1,n-2}p_{n-2,n-2} &= s_{n,n-1}p_{n,n-2} - r_{n-1,n-2}
\end{aligned}$$

Note that $s_{n,n-1}$ is a known number from the previous procedure of considering the n th row of $SP+R$. Let

$$P_{n-2} := \begin{bmatrix} p_{1,1} & p_{2,1} & \cdots & p_{n-2,1} \\ p_{1,2} & p_{2,2} & \cdots & p_{n-2,2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,n-2} & p_{2,n-2} & \cdots & p_{n-2,n-2} \end{bmatrix}$$

If $\det P_{n-2} \neq 0$, then there exist unique solutions $(s_{n-1,1}, s_{n-1,2}, \cdots, s_{n-1,n-2})$ with $s_{n-1,i} = \frac{\det P^i_{n-2}}{\det P_{n-2}}$ $\forall i = 1, 2, \cdots, n-2$ where each P^i_{n-2} is the $(n-2) \times (n-2)$ matrix of replacing the i th column of P_{n-2} by $\begin{pmatrix} s_{n,n-1}p_{n,1} - r_{n-1,1} \\ s_{n,n-1}p_{n,2} - r_{n-1,2} \\ \vdots \\ s_{n,n-1}p_{n,n-2} - r_{n-1,n-2} \end{pmatrix}$

Consider the n-2 th row of $n \times n$ matrix $SP+R$, $\cdots$

$\vdots$

Consider the third row of $n \times n$ matrix $SP+R$, let $(SP+R)_{3,i} = 0$, $i = 1, 2$. Then we have two linear equations with two variables $s_{3,1}, s_{3,2}$:

$$\begin{aligned} s_{3,1}p_{1,1} + s_{3,2}p_{2,1} + 0 \cdot p_{3,1} - s_{4,3}p_{4,1} - s_{5,3}p_{5,1} - \cdots - s_{n,3}p_{n,1} + r_{3,1} &= 0 \\ s_{3,1}p_{1,2} + s_{3,2}p_{2,2} + 0 \cdot p_{3,2} - s_{4,3}p_{4,2} - s_{5,3}p_{5,2} - \cdots - s_{n,3}p_{n,2} + r_{3,2} &= 0 \end{aligned}$$

It follows that

$$\begin{aligned} s_{3,1}p_{1,1} + s_{3,2}p_{2,1} &= s_{4,3}p_{4,1} + s_{5,3}p_{5,1} + \cdots + s_{n,3}p_{n,1} - r_{3,1} \\ s_{3,1}p_{1,2} + s_{3,2}p_{2,2} &= s_{4,3}p_{4,2} + s_{5,3}p_{5,2} + \cdots + s_{n,3}p_{n,2} - r_{3,2} \end{aligned}$$

Note that $s_{4,3}, s_{5,3}, \cdots, s_{n,3}$ are known numbers from the procedures of considering the $n, n-1, \cdots, 5, 4$ th row of matrix $SP+R$. Let

$$P_2 := \begin{bmatrix} p_{1,1} & p_{2,1} \\ p_{1,2} & p_{2,2} \end{bmatrix}$$

If $\det P_2 \neq 0$, then there exist unique solutions $(s_{3,1}, s_{3,2})$ with $s_{3,i} = \frac{\det P_2^i}{\det P_2}$, $i = 1, 2$, where each P_2^i is the 2×2 matrix of replacing the i th column of P_2 by

$$\begin{pmatrix} s_{4,3}p_{4,1} + s_{5,3}p_{5,1} + \cdots + s_{n,3}p_{n,1} - r_{3,1} \\ s_{4,3}p_{4,2} + s_{5,3}p_{5,2} + \cdots + s_{n,3}p_{n,2} - r_{3,2} \end{pmatrix}$$

Consider the second row of $n \times n$ matrix $SP + R$, let $(SP + R)_{2,1} = 0$. Then we have one linear equation with one variable $s_{2,1}$:

$$s_{2,1}p_{1,1} + 0 \cdot p_{2,1} - s_{3,2}p_{3,1} - s_{4,2}p_{4,1} - \cdots - s_{n,2}p_{n,1} + r_{2,1} = 0$$

It follows that

$$s_{2,1}p_{1,1} = s_{3,2}p_{3,1} + s_{4,2}p_{4,1} + \cdots + s_{n,2}p_{n,1} - r_{2,1}$$

Let 1×1 matrix $P_1 := (p_{1,1})$. If $\det P_1 \neq 0$, then there exists a unique solution $s_{2,1} = \frac{1}{p_{1,1}}(s_{3,2}p_{3,1} + s_{4,2}p_{4,1} + \cdots + s_{n,2}p_{n,1} - r_{2,1})$ such that $(SP + R)_{2,1} = 0$.

In summary, if $\det P_i \neq 0$, $\forall i = 1, 2, \cdots, n-1$, then there exist unique solutions $(s_{2,1}), (s_{3,1}, s_{3,2}), \cdots, (s_{n,1}, s_{n,2}, \cdots, s_{n,n-1})$ such that $SP + R$ is an $n \times n$ upper-triangular matrix. In other words, for almost all $P = (p_{ij})_{n \times n}$ and all $R = (r_{ij})_{n \times n}$, there exists a unique skew-symmetric matrix

$$S = \begin{bmatrix} 0 & -s_{2,1} & -s_{3,1} & \cdots & -s_{n-1,1} & -s_{n,1} \\ s_{2,1} & 0 & -s_{3,2} & \cdots & -s_{n-1,2} & -s_{n,2} \\ s_{3,1} & s_{3,2} & 0 & \cdots & -s_{n-1,3} & -s_{n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-1,1} & s_{n-1,2} & s_{n-1,3} & \cdots & 0 & -s_{n,n-1} \\ s_{n,1} & s_{n,2} & s_{n,3} & \cdots & s_{n,n-1} & 0 \end{bmatrix}$$

such that $SP + R$ is an $n \times n$ upper-triangular matrix. This proves the lemma.

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Vita

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